CANONICAL EXTENSIONS OF p-ADIC SHTUKAS ON TOROIDAL COMPACTIFICATIONS OF SHIMURA VARIETIES

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Abstract. We construct canonical extensions of p-adic shtukas on integral models of toroidal compactifications of abelian-type Shimura varieties with quasi-parahoric levels at any prime number p. More precisely, we define the notion of a log diamond as a v-sheaf associated with a log scheme over $\mathbb{Z}_p$ and construct a p-adic log shtuka over the log diamond of an integral toroidal compactification of an abelian-type Shimura variety by studying the "degeneration" of the shtuka at the boundary. Moreover, we provide a definition of canonical integral models of toroidal and minimal compactifications in the sense of Pappas and Rapoport, and verify it in the same generality as above.

Applications include the canonicity and functoriality of integral toroidal compactifications, as well as an axiomatic proof of the well-positionedness of all well-known stratifications on the special fiber.

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INTRODUCTION

Let (G, X) be a Shimura datum with reflex field $\mathbb{E}(G, X)$. Denote by $\{\mathrm{Sh}_K(G, X)\}_{K \subset G(\mathbb{A}_f),\mathrm{neat}}$ the associated Shimura varieties. Fix a prime number p and a place $v|p$ of $\mathbb{E}$.

In [PR24], building on Scholze's idea and theory of p-adic shtukas as substitutes for "motives" over p-adic fields, Pappas and Rapoport developed a general theory of (p-adic) shtukas and applied it to local and global Shimura varieties. They constructed canonical integral models over $\operatorname{Spec} \mathcal{O}_{\mathbb{E}_v}$, proving their uniqueness and functoriality, and treated Hodge-type Shimura varieties with stabilizer parahoric level. This was extended to general parahoric levels in [DvHKZ26], and to abelian-type Shimura varieties in [DY25] using the Kisin–Pappas–Zhou models [KPZ24].

Toroidal compactifications of Shimura varieties over the complex numbers were constructed in [AMRT10], and for mixed Shimura varieties over reflex fields in [Pin90]. For a cone decomposition Σ, the compactification $\mathrm{Sh}_K^{\Sigma}(G, X)$ has boundary charts described by certain toric embeddings of mixed Shimura varieties indexed by cusp labels $[(\Phi, \sigma)] \in \mathrm{Cusp}_K(G, X, \Sigma)$.

Integral versions of this picture were developed in [FC90], [Lan13], [MP19], and [Wu25]. In the PEL-type setting, boundary strata and their gluings are described in terms of degenerations of abelian schemes. For Shimura varieties of Hodge type and of abelian type, the strategy is different: one first obtains candidate integral models from integral toroidal compactifications of Siegel type via normalizations and passing to quotients, and the main technical difficulties arise in verifying that the boundary components of these models satisfy the required structural properties.

A natural question thus arises:

Q: If one interprets the integral models constructed in the framework of [PR24] as moduli spaces of shtukas, then for suitably defined integral models of their toroidal compactifications, one should, at least in principle, be able to provide a precise description of their degeneration behavior along the boundary.

In this paper, we discuss our understanding of this question. The goals of this paper are threefold.

1. We develop a theory of log shtukas over log diamonds over $\operatorname{Spd} \mathbb{Z}_p$, extending the theory of shtukas over diamonds in [PR24]. On the generic fiber, log shtukas correspond to certain pro-Kummer étale local systems with Hodge–Tate period maps, while on the special fiber, they recover the usual shtukas. We also prove the rigidity of extensions from the generic fiber to the integral base.
2. We construct log shtukas on toroidal compactifications of integral models of abelian-type Shimura varieties with quasi-parahoric level (which slightly generalizes Kisin–Pappas–Zhou models), and provide an axiomatic framework that is applicable to general Shimura varieties, in which the extension of shtukas to toroidal compactifications does not rely on the theory of abelian schemes or p-divisible groups. Hence, we expect that this picture should also hold in general, although canonical integral models have not yet been constructed for general parahoric groups at any p. In particular, this allows us to prove the uniqueness and functoriality of toroidal compactifications of integral models, including compatibility with changes of parahoric levels, generalizing results in [Wu25] and [Mao25a].

(3) We provide a description of how the shtukas supported on integral models degenerate along the boundary. Along the way, we show that various stratifications on the special fibers, including central leaves, Newton strata, Kottwitz–Rapoport (KR) strata, and Ekedahl–Kottwitz–Oort–Rapoport (EKOR) strata defined using shtukas have good properties along the boundary. This reproves and generalizes some results in [Box15], [LS18a] and [Mao25b].

Log shtukas on log diamonds. Let E be a local field of characteristic 0, let G be a reductive group over $\mathbb{Q}_p$, let μ be a minuscule cocharacter, and let $\mathcal{G}$ be a parahoric group scheme of G over $\mathbb{Z}_p$.

Recall that, in [PR24], a theory of shtukas over diamonds was developed. Given a scheme X that is separated, of finite type over $\mathbb{Z}_p$, there are different ways to attach certain diamonds to X, e.g., $X^{\lozenge}$, $X^{\diamond}$, and $X^{\lozenge/}$ (see [AGLR22, §2.2] and [PR24, §2.1]). One can define a family of shtukas $(\mathscr{P}, \phi_{\mathscr{P}})$ on $X^{\lozenge}$ (with one leg bounded by μ) as a 1-morphism between v-stacks $X^{\lozenge} \to \mathrm{Sht}_{\mathcal{G},\mu}$. For a locally noetherian adic space X over $\mathrm{Spa}(E, \mathcal{O}_E)$, the authors proved that there is an equivalence between $\mathcal{G}$-shtukas over $X^{\lozenge}$ and pairs $(\mathbb{P}, H)$ consisting of pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors $\mathbb{P}$ over $X^{\lozenge}$ and Hodge-Tate maps H (see [PR24, Prop. 2.5.3]). An important feature is that, given $\mathscr{X}$ a separated normal scheme of finite type over $\mathcal{O}_E$, a morphism of $\mathcal{G}$-shtukas on the generic fiber $(\mathscr{X}_\eta)^{\lozenge}$ extends uniquely over the integral base $\mathscr{X}^{\lozenge}$ (see [PR24, Thm. 2.7.7 and Cor. 2.7.9]). In characteristic p, shtukas are crystalline in nature (see [PR24, Thm. 2.3.8 and Ex. 2.4.9]).

To define the extension of shtukas on toroidal compactifications, we need to generalize the theory of diamonds and shtukas to the logarithmic setting. Let $(X, \mathcal{M}_X)$ be a locally Noetherian fs log adic space over $\mathrm{Spa}(\mathbb{Z}_p)$. Recall that in [DLLZ23a], a log structure $(\mathcal{M}_X, \alpha_X)$ on X is a pair consisting of $\mathcal{M}_X$, a sheaf of monoids over $X_{\text{ét}}$, together with a morphism of sheaves of monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}}$ such that $\alpha_X^{-1}(\mathcal{O}_{X_{\text{ét}}}^\times) \to \mathcal{O}_{X_{\text{ét}}}^\times$ is an isomorphism.

Recall that the diamond $X^{\lozenge}$ associated with X is a v-sheaf, and we should associate a v-sheafy diamond with the log adic space $(X, \mathcal{M}_X)$ while preserving the log-structure information coming from the étale topology.

Definition A (Definition 2.13)**.** *Let* $S \in \mathrm{Perf}$*. Then* $(X, \mathcal{M}_X)^{\lozenge}(S)$ *consists of isomorphism classes of pairs* $((S^\sharp, \mathcal{M}_{S^\sharp}), f)$*, where* $S^\sharp$ *is an untilt of* S*,* $\mathcal{M}_{S^\sharp}$ *is a* saturated and fine perfectoid *log structure on* $S^\sharp$*, and* $f : (S^\sharp, \mathcal{M}_{S^\sharp}) \to (X, \mathcal{M}_X)$ *is a morphism between log adic spaces.*

Here the adjective "*fine perfectoid*" will be introduced in Definition 2.6. Roughly speaking, we require that $\mathcal{M}_{S^\sharp}$ has local charts that are uniquely p-divisible and are "perfections" of finitely generated monoids in an appropriate sense. We show that

Theorem A (Theorem 2.18)**.** $(X, \mathcal{M}_X)^{\lozenge}$ *is a* v*-sheaf on* Perf*.*

For separated schemes of finite type, one can define log diamonds $(X, \mathcal{M}_X)^{\lozenge}$, $(X, \mathcal{M}_X)^{\diamond}$, and $(X, \mathcal{M}_X)^{\lozenge/}$ as v-sheaves in a similar way.

Definition B (Definition 2.33)**.** *The groupoid of log* $\mathcal{G}$*-shtukas on* $(X, \mathcal{M}_X)^{\lozenge}$ *(with one leg bounded by* μ*) is defined as (see Definition A.3)*

$$\mathrm{Sht}_{\mathcal{G},\mu}(X, \mathcal{M}_X) := \underset{(S^\sharp, \mathcal{M}_{S^\sharp}, f_{S^\sharp}) \in ((X, \mathcal{M}_X)^{\lozenge})^{\mathrm{op}}}{2\text{-}\lim} \mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp).$$

In other words, giving a log shtuka amounts to giving a 1-morphism between v-stacks

$$(X, \mathcal{M}_X)^{\lozenge} \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

Let us now focus on the generic fiber. In this situation, X is a locally Noetherian fs log adic space over $\mathrm{Spa}\,\mathbb{Q}_p$. We consider the category of pro-Kummer-étale $\mathbb{Z}_p$-local systems on X, which is the generalization of the category of pro-étale $\mathbb{Z}_p$-local systems to the log case. To formulate an equivalence of categories, we need to restrict the notion to a so-called *pro-p-Kummer-étale* $\mathbb{Z}_p$-local system, which requires that the associated representation from the Kummer étale fundamental group $\pi_1^{\text{két}}(X,\zeta)$ of $(X,\mathcal{M}_X)$ at each log geometric point ζ has no contribution from the prime-to-p part; see Definition 2.42. In particular, when the local system has unipotent monodromy along the boundaries, it is pro-p-Kummer.

Theorem B (Theorem 2.49)**.** *Let* $(X,\mathcal{M}_X)$ *be a locally Noetherian fs log adic space over* $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$*, and let* $(\mathcal{G},\mu)$ *be as above. Then there exists an equivalence of categories between:*

1. *log* $\mathcal{G}$*-shtukas over* $(X,\mathcal{M}_X)^\Diamond \to \mathrm{Spd}\, E$ *with one leg bounded by* μ*;*
2. *pairs* $(\mathbb{P},H)$ *where* $\mathbb{P}$ *is a pro-p-Kummer-étale* $\underline{\mathcal{G}(\mathbb{Z}_p)}$*-torsor defined over* $X^{\log\Diamond}$ *and* $H:\mathbb{P}\to\mathcal{F}^\Diamond_{G,\mu^{-1}}$ *is a* $\underline{\mathcal{G}(\mathbb{Z}_p)}$*-equivariant map of v-sheaves over* $\mathrm{Spd}\, E$*.*

For a related theorem in the log prismatic theory, see [KY25, Thm. 7.36] and an update [IKY26] by Inoue-Koshikawa-Yao. In fact, our proof is similar to the one presented there. This theorem also tells us that $(\mathrm{Sh}^\Sigma_K)^{\log\Diamond}\to\mathrm{Sht}_{\mathcal{G}^c,\mu^c}$ should not factor through $\mathrm{Sh}^{\min,\Diamond}_K$, even though the Hodge-Tate period map on $\mathrm{Sh}^{\Sigma,\Diamond}_{K^p}$ should factor through $\mathrm{Sh}^{\min,\Diamond}_{K^p}$.

Now, let $(X,\mathcal{M}_X)$ be an fs log scheme where X is a separated, normal, and flat scheme of finite type over $\mathrm{Spec}\,\mathbb{Z}_p$. There are different ways to associate a log adic space to $(X,\mathcal{M}_X)$ and produce diamonds $(X,\mathcal{M}_X)^\Diamond$, $(X,\mathcal{M}_X)^\diamond$, $(X,\mathcal{M}_X)^{\Diamond/}$. The log generalization of [PR24, Thm. 2.7.7] is as follows:

Theorem C (Theorem 2.55)**.** *The restriction functor*

$$\mathrm{Res}^{\mathscr{X}}_{\mathscr{X}_\eta}:\mathrm{Sht}_{\mathcal{G},\mu}((\mathscr{X},\mathcal{M}_{\mathscr{X}})^{\Diamond/})\longrightarrow\mathrm{Sht}_{\mathcal{G},\mu}((\mathscr{X}_\eta,\mathcal{M}_{\mathscr{X}_\eta})^{\Diamond})$$

is fully faithful.

In addition, we show that log shtukas on special fibers are non-log shtukas (see Corollary 2.58). That is, in characteristic p, a log $\mathcal{G}$-shtuka (with one leg bounded by μ) on $(\mathscr{X},\mathcal{M}_{\mathscr{X}})^\diamond$ is equivalent to an actual $\mathcal{G}$-shtuka (with one leg bounded by μ) on $\mathscr{X}^\diamond$.

Canonical integral models. Let (G,X) be a Shimura datum. Fix an open compact subgroup $K\subset G(\mathbb{A}_f)$. Choose an admissible (rational polyhedral), smooth, projective cone decomposition Σ (without self-intersections). Denote the toroidal compactification by $\mathrm{Sh}^\Sigma_K:=\mathrm{Sh}^\Sigma_K(G,X)$ and the minimal compactification by $\mathrm{Sh}^{\min}_K:=\mathrm{Sh}^{\min}_K(G,X)$; they are defined over the reflex field $\mathbb{E}$. There is a proper morphism $\oint^\Sigma_K:\mathrm{Sh}^\Sigma_K\to\mathrm{Sh}^{\min}_K$ that is compatible with the stratifications on the source and the target (see [AMRT10] and [Pin90]).

Fix a prime number p and a place v of $\mathbb{E}$ over p. Let $E=\mathbb{E}_v$ be the completion. Following [LS18a, Prop. 2.1.2] and [MP19, Thm. 4.1.5], one expects that there exist normal, proper, flat models $\mathscr{S}^\Sigma_K$ and $\mathscr{S}^{\min}_K$ for $\mathrm{Sh}^\Sigma_{K,E}$ and $\mathrm{Sh}^{\min}_{K,E}$ over $\mathcal{O}_E$, respectively, such that a list of qualitative properties stated in Axiom 5.1 is satisfied.

Let us explain some of the axioms. Fix a cusp label $[\Phi]=[(Q_\Phi,X^+_\Phi,g_\Phi)]\in\mathrm{Cusp}_K(G,X)$. We have a tower of integral models of mixed Shimura varieties:

$$\mathscr{S}_{K_\Phi}\to\mathscr{S}_{\overline{K}_\Phi}\to\mathscr{S}_{K_{\Phi,h}}\quad(:=\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)\to\mathscr{S}_{\overline{K}_\Phi}(\overline{P}_\Phi,\overline{D}_\Phi)\to\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h}))$$

which are normal flat schemes of finite type over $\mathcal{O}_E$ that extend those on the generic fiber. The first map is also a torsor under the same split torus $\mathbf{E}_{K_\Phi}$. The second map is proper and surjective. There is a $\Delta_{\Phi,K}$-action on the whole tower that factors through a finite quotient on $\mathscr{S}_{K_{\Phi,h}}$.

Pick $\sigma\in\Sigma^+(\Phi)\subset X_*(\mathbf{E}_{K_\Phi})\otimes\mathbb{R}$. There is a normal subgroup $\Delta^\circ_{\Phi,K}\lhd\Delta_{\Phi,K}$ that stabilizes σ. The boundary of $\mathscr{S}^\Sigma_K$ (resp. $\mathscr{S}^{\min}_K$) is stratified by $\mathscr{Z}_{[(\Phi,\sigma)],K}\cong\Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi,\sigma}$ (resp. $\mathscr{Z}_{[\Phi],K}\cong\Delta_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}$). Étale locally on $\mathscr{S}^\Sigma_K(G,X)$ at the boundary stratum $\mathscr{Z}_{[(\Phi,\sigma)],K}$, the morphism $\mathscr{S}_K\hookrightarrow\mathscr{S}^\Sigma_K$ is a toroidal embedding $\Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}\hookrightarrow\Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(\sigma)$.

Moreover, there is a stronger strata-preserving isomorphism

$$\mathfrak{X}^{\circ}_{\sigma,K} \cong \Delta^{\circ}_{\Phi,K}\backslash(\mathscr{S}_{K_\Phi}(\sigma))^{\wedge}_{\mathscr{S}^{+}_{K_\Phi,\sigma}}, \quad \mathscr{S}^{+}_{K_\Phi,\sigma} := \cup_{\tau\subset\sigma,\tau\in\Sigma^{+}(\Phi)}\mathscr{S}_{K_\Phi,\tau},$$

where

$$\mathfrak{X}^{\circ}_{\sigma,K} := (\mathscr{S}^{\Sigma}_{K})^{\wedge}_{\mathcal{Z}^{+}_{\sigma,K}}, \quad \mathcal{Z}^{+}_{\sigma,K} := \cup_{\tau\subset\sigma,\tau\in\Sigma^{+}(\Phi)}\mathcal{Z}_{[(\Phi,\tau)],K}.$$

Let (G, X) be an abelian-type Shimura datum, let p be any prime, and let $K = K_pK^p$ be any neat level. By a series of works [FC90], [Lan16], [MP19], [Wu25], and others, there exist good toroidal and minimal compactifications of integral models satisfying Axiom 5.1 (see Theorem 5.5).

On the other hand, in [PR24], [DvHKZ26], and [DY25], when K_p is parahoric, a conjectural framework of *canonical integral models* of the pro-system $\{\mathrm{Sh}_{K_pK^p}(G, X)\}_{K^p\subset G(\mathbb{A}^p_f)}$ was also introduced and constructed essentially in the abelian-type case.

In order to define and construct *canonical integral models* of toroidal compactifications, we combine the two above-mentioned aspects and study the canonical integral models of *boundary mixed* Shimura varieties. For a mixed Shimura datum $(P, \mathcal{X})$, P is a non-reductive group. We generalize the notions of local models and shtukas with one leg bounded by μ to non-reductive groups, and prove many functoriality results that are used in later sections. The definitions and propositions are parallel to those defined using reductive groups.

- We formulate a conjectural framework 4.2 for *canonical integral models* of the pro-system of mixed Shimura varieties arising from the boundary $\{\mathrm{Sh}_{K_{\Phi,p}K^p_\Phi}(P_\Phi, D_\Phi)\}_{K^p_\Phi\subset P_\Phi(\mathbb{A}^p_f)}$, following [PR24, Conj. 4.2.2], [DvHKZ26, Def. 4.1.2], and [DY25, Def. 4.3]. In particular, we require an extension of the $\mathcal{P}^*_\Phi$-shtuka with one leg bounded by μ^*_Φ from $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$ to the integral model $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)$. Here we use a slightly different group P^*_Φ than P^c_Φ; see Remark 4.4. We show the uniqueness and functoriality of such canonical integral models; see Proposition 4.10 and Corollary 4.11.
- We formulate a conjectural framework for *canonical integral models* of the pro-system of toroidal compactifications $\{\mathscr{S}^{\Sigma}_K\}_{K^p}$ and minimal compactifications $\{\mathscr{S}^{\mathrm{min}}_K\}_{K^p}$ (see Definition 5.27). We show the uniqueness and functoriality of such canonical integral models (see Proposition 5.35). We also obtain analogous statements for integral models of minimal compactifications.

Theorem D (Theorem 5.28 and Theorem 6.26)**.** *Let (G, X) be an abelian-type Shimura datum, p be any prime, and K_p be any quasi-parahoric level.*

(1) There exist canonical integral models $\{\mathscr{S}_K(G, X)\}_{K^p}$.
(2) There exist canonical integral models $\{\mathscr{S}_{K_{\Phi,p}K^p_\Phi}(P_\Phi, D_\Phi)\}_{K^p_\Phi\subset P_\Phi(\mathbb{A}^p_f)}$ for each $[\Phi]$.
(3) There exist canonical integral models $\{\mathscr{S}^{\Sigma}_K\}_{K^p}$ for a final collection of smooth projective cone decompositions Σ, and canonical integral models $\{\mathscr{S}^{\mathrm{min}}_K\}_{K^p}$.

Such canonical integral models are unique and functorial.

Remark. *We remark that, even for compact Shimura varieties, the first item already extends the known cases in the literature (see* [PR24], [DvHKZ26] *and* [DY25]*) to cases allowing all primes and all quasi-parahoric levels for all abelian-type Shimura data. In* [DvHKZ26]*, the authors showed the first item of Theorem D in the Hodge-type case for any p and any quasi-parahoric K_p; in* [DY25]*, the authors proved the result in the abelian-type case when $p > 2$ and K_p is parahoric.*

In fact, even if one just wants to generalize these results to compactifications, it is still much more convenient to work with Bruhat-Tits stabilizer levels than parahoric levels. This fact forced us to prove the theorem in the above generality.

Remark (See also Remark 7.25)**.** *To show the theorem above, we have to consider non-R-smooth (cf.* [KPZ24, 2.1.4] *for the definition of R-smoothness) Shimura data (G, X) with parahoric levels K°_p whose intersections with $G^{\mathrm{der}}(\mathbb{Q}_p)$ are* not *parahoric. For this, we need to construct Hodge-type* *liftings that are* ***accessible*** *(see Definition 6.17) to the abelian-type ones in the quasisplit nonsplit* $D^{\mathbb{H}}$*-type case at all primes. This calls for a certain group-theoretic refinement in the construction.*

However, our construction itself cannot deduce new cases of local model diagrams, which is a more difficult problem. When $p > 2$*, this has been established in* [KPZ24]*; when* $p = 2$*, for the progress in this direction, cf. Jie Yang's work* [Yan25]*. One can show that, with inputs in* [DvHKZ26]*, the* $\mathcal{G}^{\mathrm{ad},\circ}$*-local model diagrams in* [KPZ24] *are schematic local model diagrams in the sense of* [PR24, Def. 4.9.1]*; for this, see Proposition 7.22.*

An essential feature of canonical integral models of compactifications is the existence of a log $\mathcal{G}^c$-shtuka $\mathscr{P}^{\mathrm{can}}$ with one log bounded by μ on $\mathscr{S}_K^\Sigma$ that extends the given shtuka on $\mathscr{S}_K$; that is, for a canonical integral model $\mathscr{S}_K^\Sigma$, there is a morphism

$$(\mathscr{S}_K^\Sigma)^{\log \Diamond} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c}.$$

Our method here provides a description of these extensions at boundaries, which will be given below in more detail; and an important advantage is that the extension step only uses (log) $\mathcal{G}^c$-shtukas, which makes it possible to be applied to cases beyond abelian-type Shimura varieties (see Theorem 5.19 and Lemma 5.15).

Shtukas at the boundary. From this subsection onward, fix any Shimura datum (G, X), and let K_p be a quasi-parahoric level subgroup. We work with canonical integral models $\{\mathscr{S}_K^\Sigma\}_{K^p}$ and $\{\mathscr{S}_K^{\min}\}_{K^p}$, and we prove many results without the use of abelian schemes and p-divisible groups[1]. This framework allows us to work with more general types of Shimura varieties once certain axioms can be verified.

Let $\mathfrak{W} = \mathrm{Spf}(R, I) \subset \mathfrak{X}_\sigma^\circ$ be an affine open formal subscheme. We can consider the flat morphisms $W = \mathrm{Spec}\, R \to \mathscr{S}_K^\Sigma$ and $W \to \Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_\Phi}(\sigma)$. Let $W^0 \subset W$ be the common open subscheme associated with $\mathscr{S}_K \subset \mathscr{S}_K^\Sigma$ and $\Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_\Phi} \subset \Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_\Phi}(\sigma)$.

Proposition (Proposition 5.34)**.** *We have the following important diagram:*

$$\begin{tikzcd} \mathscr{S}_K(G,X)^\diamond \arrow[d] & W^{0,\diamond} \arrow[l] \arrow[r] & \Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^\diamond \arrow[r] \arrow[d] & \Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^\diamond \arrow[d] \\ \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1} & & \mathrm{Sht}_{\mathcal{P}_\Phi^*,\mu_\Phi^*,\delta=1} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & \mathrm{Sht}_{\mathcal{G}_{\Phi,h}^*,\mu_{\Phi,h}^*,\delta=1}. \end{tikzcd} \tag{0.1}$$

This tells us how shtukas degenerate along the boundary. One should compare this result with the degeneration of abelian schemes with extra structures introduced in [FC90] and [Lan13].

On the other hand, we can extend the shtukas over the boundary. Constructing log shtukas on $(\mathscr{S}_{K_\Phi}(\sigma), \mathcal{M}_{\mathscr{S}_{K_\Phi}(\sigma)})$ is more approachable than constructing log shtukas on $(\mathscr{S}_K^\Sigma, \mathcal{M}_{\mathscr{S}_K^\Sigma})$, since we can work with an actual torsor under a split torus. By gluing log shtukas on $(\mathscr{S}_{K_\Phi}(\sigma), \mathcal{M}_{\mathscr{S}_{K_\Phi}(\sigma)})$ for $[(\Phi, \sigma)] \in \mathrm{Cusp}_K(G, X, \Sigma)$, we get a log shtuka on $(\mathscr{S}_K^\Sigma, \mathcal{M}_{\mathscr{S}_K^\Sigma})$:

Theorem E (Corollary 5.20, Theorem 5.29)**.**

(1) For each $[(\Phi, \sigma)] \in \mathrm{Cusp}_K(G, X, \Sigma)$*, there is a unique log* $\mathcal{P}_\Phi^*$*-shtuka on* $\Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_\Phi}(\sigma)$ *with one leg bounded by* μ_Φ^* *extending the one on the generic fiber, i.e., there exists a morphism* $\mathscr{S}_{K_\Phi}(\sigma)^{\log \Diamond/} \to \mathrm{Sht}_{\mathcal{P}_\Phi^*,\mu_\Phi^*,\delta=1}$.

(2) There exists a unique log $\mathcal{G}^c$*-shtuka on* $\mathscr{S}_K^\Sigma$ *with one leg bounded by* μ^c *extending the one on the generic fiber, i.e., there exists a morphism* $(\mathscr{S}_K^\Sigma)^{\log \Diamond/} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$.

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[1] In fact, our method works in a more general context; see the discussion in §5.4; indeed, we only need a weaker assumption 5.26.

Since $\mathscr{S}_K^\Sigma(G,X)$ is proper, we have $(\mathscr{S}_K^\Sigma, \mathcal{M}_{\mathscr{S}_K^\Sigma})^{\lozenge/} = (\mathscr{S}_K^\Sigma, \mathcal{M}_{\mathscr{S}_K^\Sigma})^{\lozenge} = (\mathscr{S}_K^\Sigma, \mathcal{M}_{\mathscr{S}_K^\Sigma})^{\diamond}$. We restrict it to the special fiber and obtain a morphism $(\mathscr{S}_{K,\bar{s}}^\Sigma, \mathcal{M}_{\mathscr{S}_{K,\bar{s}}^\Sigma})^{\diamond} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c}$, where $\bar{s} = \operatorname{Spec} \overline{\mathbb{F}}_p$. Let $\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c}$ be the Witt vector $\mathcal{G}^c$-shtuka with one leg bounded by μ^c introduced in [XZ17] (cf. [SYZ21]; see [DvHKZ24, Rmk. 3.1.8] for the sign convention). Combining with Corollary 2.58, we obtain:

Corollary A (Corollary 5.30)**.** *We have a $\mathcal{G}^c$-shtuka with one leg bounded by μ^c on the special fiber of $\mathscr{S}_K^\Sigma(G,X)$, i.e., a morphism $\mathscr{S}_K^\Sigma(G,X)_{\bar{s}}^{perf} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1}$.*

Corollary B (Corollary 5.31)**.** *We have a $\mathcal{G}^c$-shtuka with one leg bounded by μ^c on the big diamond $\mathscr{S}_K(G,X)^{\lozenge}$, i.e., a morphism $\mathscr{S}_K(G,X)^{\lozenge} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$.*

Neither Corollary A nor Corollary B appears previously in the literature, even in cases such as modular curves. For cases with good moduli interpretations, we expect that these results could be proved using log p-divisible groups.

Well-positioned subschemes. Using the Witt vector shtuka $\mathscr{S}_K(G,X)_{\bar{s}}^{\mathrm{perf}} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1}$, and following [PR24], we can construct Newton strata, central leaves, KR strata, and EKOR strata on $\mathscr{S}_K(G,X)_{\bar{s}}$.

In [Box15] and [LS18a], the authors formulate the notion of *well-positioned subschemes*. Well-positioned strata have good (partial) toroidal and minimal compactifications along the boundary, with many good qualitative properties described in Axiom 5.1, as if they were Shimura varieties in characteristic 0. It was proved in [LS18a] that, for PEL-type Shimura varieties, Newton strata, central leaves, KR strata, and EKOR strata are well positioned. These results were generalized to the Hodge-type case in [Mao25b], where it was shown that their partial minimal compactifications are again of the same type of strata (e.g., the boundary of the partial minimal compactification of a Newton stratum is stratified by Newton strata on the boundary). The proofs in [LS18a] essentially used the degeneration of p-divisible groups along the boundary. Under the degeneration diagram (0.1) (cf. Proposition 5.34), and since the shtukas in characteristic p are crystalline in nature, we find that the well-positioned property of all these strata can be naturally explained using the degeneration of shtukas.

Assume there exist canonical integral models $\{\mathscr{S}_K^\Sigma\}_{K^p}$ and $\{\mathscr{S}_K^{\min}\}_{K^p}$ in the sense of Definition 5.27 (for example, in the setting of Theorem D).

Theorem F.

1. *Newton strata (resp. central leaves, KR strata, and EKOR strata), their connected components, and their closures are well positioned locally closed subschemes in the sense of Definition 7.1.*
2. *The boundary of the partial minimal compactifications of Newton strata (resp. central leaves, KR strata) is stratified by Newton strata (resp. central leaves, KR strata) on the boundary; see Proposition 7.7, 7.12, and 7.19 for details.*

Besides generalizing the well-positioned property and the boundary descriptions of partial minimal compactifications from the Hodge-type case to a more general framework (at least to the abelian-type case), we also provide a new description of partial toroidal compactifications of these strata.

Proposition (Proposition 7.13, 7.14, 7.21, 7.26)**.** *The partial toroidal compactifications of Newton strata (resp. central leaves, KR strata, and EKOR strata) coincide with the Newton strata (resp. central leaves, KR strata, and EKOR strata) defined using $\mathscr{S}_K^\Sigma(G,X)_{\bar{s}}^{perf} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1}$.*

Remark. *In a recent paper [Ino25], Kentaro Inoue constructed a log prismatic realization in the Hodge-type case for smooth integral models of toroidal compactifications by developing a log prismatic Dieudonné theory using log p-divisible groups. Combining with it, the method provided here can be* *extended to the log prismatic context for more general* (G, X)*, and may also give the log prismatic realizations finer boundary descriptions. In fact, we can construct a realization functor from log prismatic* F*-crystals to log shtukas that is compatible with the étale realization functor. Together with Inoue, we plan to explore this direction.*

Structure of the article. Section 1 can be viewed as a preliminary section of this paper, where we start by generalizing the definition of some objects in [SW20] to non-reductive groups and proving basic properties. In Section 2, we define and study log diamonds and shtukas, generalizing and modifying the previous work [PR24] and [KY25]. Some results analogous to [PR24, Sec. 2] will be shown in the log setting. In Section 3, we study the p-adic local systems on mixed Shimura varieties. In Section 4, we define canonical integral models for boundary mixed Shimura varieties. Some basic properties of these canonical integral models will be proved. In Section 5, we show the existence of canonical extensions of shtukas (or rather, log shtukas) in an axiomatic setup. We also define canonical integral models for toroidal/minimal compactifications. In Section 6, we demonstrate that canonical integral models for compactifications exist for all abelian-type Shimura data, all primes, and all quasi-parahoric levels. In particular, the canonical integral models for (boundary mixed) Shimura varieties also exist in the same generality. In Section 7, we prove the well-positionedness of all well-known stratifications of special fibers of Shimura varieties under the axioms in Section 5.

Here is a workflow diagram:

$$\begin{tikzcd}
\boxed{\S 2} \arrow[rrr] & & & \boxed{\S 5} \arrow[r] & \boxed{\S 7} \\
\boxed{\S 1} \arrow[r] & \boxed{\S 3} \arrow[r] & \boxed{\S 4} \arrow[r] \arrow[ur] & \boxed{\S 6}. \arrow[u] &
\end{tikzcd}$$

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Notation and conventions. Fix a prime number $p > 0$. All monoids are commutative. All rings have identities. For the conventions in p-adic geometry (and perfectoid geometry), we follow [SW20] and [Sch26]. For the definition of an adic space being *locally Noetherian*, we follow [DLLZ23a, p.4]. For conventions on (mixed) Shimura varieties and compactifications, we mainly follow [Wu25] (we refer readers to the latest version here, which will be updated on arXiv in the future). For the conventions in log geometry, we mainly follow [Kat89], [Ogu18], and [DLLZ23a]. A constant sheaf of monoids on X with values in P is denoted by P_X instead of $\underline{\mathsf{P}}$ to avoid awkward conventions such as "$\underline{\mathsf{P}}^{a}$". Log structures are written multiplicatively, but charts are written additively. For a Shimura datum (G, X), G is a reductive group over $\mathbb{Q}$. We also write $G \otimes_{\mathbb{Q}} \mathbb{Q}_p$ as G when it is clear in the context. There are several "E" in different fonts: In general, "$\mathbb{E}$" denotes global fields; "E" denotes local fields; "$\mathbf{E}$" denotes tori. For a smooth affine group scheme $\mathcal{G}$ over $\mathbb{Z}_p$, we usually denote $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $\breve{K}_p = \mathcal{G}(\breve{\mathbb{Z}}_p)$. Here $(\breve{*})$ means the set of $\breve{\mathbb{Z}}_p$-points. Let Perf denote the category of perfectoid spaces of characteristic p, and Perfd denote the category of all perfectoid spaces. Let $\mathrm{PCAlg}^{\mathrm{op}}$ denote the category of affine perfect schemes. Some Galois groups: $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$, $I = \mathrm{Gal}(\overline{\mathbb{Q}}_p|\breve{\mathbb{Q}}_p)$, $\Sigma_0 = \langle \sigma \rangle = \mathrm{Gal}(\breve{\mathbb{Q}}_p|\mathbb{Q}_p)$.

The group-theoretic conventions in §1 and a few lemmas and definitions in §3 and §4 are slightly different from other places in the article. In §1, G usually denotes the Levi quotient of a linear algebraic group P, while an embedding of P into a reductive group is denoted by $P \hookrightarrow G'$. In other places, G usually denotes the group that P (or rather, P_{Φ}) maps into.

1. SHTUKAS FOR NON-REDUCTIVE GROUPS

To understand the geometry of toroidal compactifications of Shimura varieties, we study mixed Shimura varieties. As a first step, we discuss local models and shtukas for non-reductive groups in detail. Many properties are analogous to those for reductive groups; consequently, we present only the properties needed in later sections.

1.1. The B_{dR}^+-affine Grassmannian.

1.1.1. *Setup.* Let $\mathcal{P}$ be a smooth affine group scheme over $\mathbb{Z}_p$.

Definition/Proposition 1.1 ([SW20, §19, 20])**.** *Recall that the B_{dR}^+-affine Grassmannian $\mathrm{Gr}_{\mathcal{P}} \to \mathrm{Spd}\,\mathbb{Z}_p$ is a v-sheaf on* Perf *admitting the following equivalent descriptions.*

1. *It is the étale sheafification of the presheaf quotient $L\mathcal{P}/L^+\mathcal{P}$. Recall that the loop group functor $L\mathcal{P}$ and the positive loop group subfunctor $L^+\mathcal{P}$ are defined on $\mathrm{Spd}\,\mathbb{Z}_p$: Given $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ equipped with an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over $\mathrm{Spa}(\mathbb{Z}_p)$, we have*
$$L\mathcal{P} : (R^\sharp, R^{\sharp+}) \mapsto \mathcal{P}(B_{\mathrm{dR}}(R^\sharp)), \quad L^+\mathcal{P} : (R^\sharp, R^{\sharp+}) \mapsto \mathcal{P}(B_{\mathrm{dR}}^+(R^\sharp)).$$
2. *It is a functor whose $S = \mathrm{Spa}(R, R^+)$-points parametrize untilts $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over $\mathrm{Spa}(\mathbb{Z}_p)$ together with a $\mathcal{P}$-torsor $\mathcal{E}$ on $\mathrm{Spec}\,B_{\mathrm{dR}}^+(R^\sharp)$ and a trivialization of it over $\mathrm{Spec}\,B_{\mathrm{dR}}(R^\sharp)$.*
3. *It is a functor whose $S = \mathrm{Spa}(R, R^+)$-points parametrize untilts $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over $\mathrm{Spa}(\mathbb{Z}_p)$ together with a $\mathcal{P}$-torsor $\mathcal{E}$ on $S\dot{\times}\mathbb{Z}_p$ and a trivialization of $\mathcal{E}|_{S\dot{\times}\mathbb{Z}_p\setminus S^\sharp}$ that is meromorphic along $S^\sharp$.*

Proof. The equivalence of the second and third descriptions is proved using Beauville–Laszlo gluing, which is compatible with the Tannakian formalism; see, for example, [Ans22, Lem. 7.2]. The equivalence of the first and second descriptions is given by [SW20, Prop. 19.1.2, 19.5.3]. The proofs do not need the group to be reductive. □

We recall some simple facts.

Lemma 1.2 ([PR08, Thm. 1.4])**.** *Let $\mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of smooth affine group schemes. Assume that $\mathcal{P}_1$ is the smoothing of the closure of P_1 in $\mathcal{P}_2$. If $\mathcal{P}_2/\mathcal{P}_1$ is affine, then $\mathrm{Gr}_{\mathcal{P}_1} \to \mathrm{Gr}_{\mathcal{P}_2}$ is a closed embedding.*

Proof. This is essentially [PR08, Thm. 1.4] (cf. [SW20, Lem. 19.1.5], where we only need the input that $\mathcal{P}_2/\mathcal{P}_1$ is affine). Note that the proof in [SW20, Lem. 19.1.5] does not require that $\mathcal{P}_1 \to \mathcal{P}_2$ is a closed embedding; it only requires that, for any $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ with an untilt $\mathrm{Spa}(R^\sharp, R^{\sharp+})$, we have $\mathcal{P}_1(B_{\mathrm{dR}}^+(R^\sharp)) = P_1(B_{\mathrm{dR}}(R^\sharp)) \cap \mathcal{P}_2(B_{\mathrm{dR}}^+(R^\sharp))$. This is always true under our assumption; see [PR24, Lem. 4.6.1]. □

Lemma 1.3. *Let $\mathcal{X}$ be an affine scheme of finite type over $\mathbb{Z}_p$, and let $\mathcal{X} \hookrightarrow \mathbb{A}^n$ be a closed embedding. Then the natural inclusion functor $L^+\mathcal{X} \to L^+\mathbb{A}^n$, as well as $L\mathcal{X} \to L\mathbb{A}^n$, is a closed embedding. Moreover, $L^+\mathcal{X} \to L\mathcal{X}$ is a closed embedding.*

Proof. We show that $L\mathcal{X} \to L\mathbb{A}^n$ is a closed embedding. In the equal-characteristic case over geometric points, see the descriptions in [PR08, §1]. In the mixed-characteristic case, see [Zhu17, Prop. 1.1]. In general, given an $S = \mathrm{Spa}(R, R^+)$-point $x \in L\mathbb{A}^n(R, R^+) = B_{\mathrm{dR}}(R^{\sharp+})^n$, we need to prove that the locus where this point lies in the image of $L\mathcal{X}$ is closed. The result follows from the proof of [SW20, Lem. 19.1.4]: by induction, assume $x \in B_{\mathrm{dR}}^+(R^{\sharp+})^n/\xi B_{\mathrm{dR}}^+(R^{\sharp+})^n$; then x lies in the image of $L\mathcal{X}$ if and only if its reduction modulo ξ lies in the image of $\mathcal{X}$. Similarly, $L^+\mathcal{X} \to L^+\mathbb{A}^n$ is a closed embedding, and therefore $L^+\mathcal{X} \to L\mathcal{X}$ is a closed embedding. □

Also, let us introduce a useful lemma. Given a positive integer N, recall that we have a notion of congruence quotient group $L^{\leq N}\mathcal{P} = L^+\mathcal{P}/L_N^+\mathcal{P}$, where $L_N^+\mathcal{P} := \ker(L^+\mathcal{P} \to \mathcal{P}(B_{\mathrm{dR}}^+/\xi^N))$.

Lemma 1.4. *Let $X \subset \mathrm{Gr}_{\mathcal{P}}$ be a quasi-compact and quasi-separated sub-sheaf, then the natural action $a_X : L^+\mathcal{P} \times X \to \mathrm{Gr}_{\mathcal{P}}$ factors through $a_{X,N} : L^{\leq N}\mathcal{P} \times X \to \mathrm{Gr}_{\mathcal{P}}$ for some large N. Moreover, $a_{X,N}$ is proper.*

Proof. Let us show the first statement. Fix a faithful representation $\rho : \mathcal{P} \to \mathrm{GL}_n$ such that $\mathrm{GL}_n / \mathcal{P}$ is quasi-affine (such an embedding exists; see [PR08, Prop. 1.3]; see also [Ric19, Cor. 3.8] for a more general statement), and then apply the arguments in [SW20, Lem. 19.1.5] (cf. [Zhu17, Prop. 1.20]) to show that $\mathrm{Gr}_{\mathcal{P}} \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a locally closed embedding. The image of X in $\mathrm{Gr}_{\mathrm{GL}_n}$ is quasi-compact and quasi-separated, and $L_N^+\mathcal{P} = L^+\mathcal{P} \cap L_N^+ \mathrm{GL}_n$. It suffices to work with $\mathcal{P} = \mathrm{GL}_n$, where the result is well-known. Now, $a_{X,N}$ is quasi-compact since both $L^{\leq N}\mathcal{P}$ and X are quasi-compact, is partially proper since $L^{\leq N}\mathcal{P}$, X and $\mathrm{Gr}_{\mathcal{P}}$ are partially proper by construction. In particular, $a_{X,N}$ is proper by [SW20, Cor. 17.4.8]. ☐

1.1.2. *Diamond functors.* For a scheme X over $\mathbb{Z}_p$, there are three different notions of diamonds associated with X. See [AGLR22, §2.2] and [PR24, §2.1].

- "big diamond" $X^{\Diamond}$: A v-sheaf $X^{\Diamond}$ sending any affinoid perfectoid $S \in \mathrm{Ob}\,\mathrm{Perf}$ to the isomorphism classes of tuples $(S^\sharp, \iota, f)$, where $(S^\sharp, \iota)$ is an untilt and $(f : S^\sharp \to X) \in X(S^\sharp)$.
- "small diamond" $X^{\diamond}$: A v-sheaf $X^{\diamond}$ sending any affinoid perfectoid $S = \mathrm{Spa}(A, A^+) \in \mathrm{Ob}\,\mathrm{Perf}$ to the isomorphism classes of tuples $(S^\sharp = \mathrm{Spa}(A^\sharp, A^{\sharp,+}), \iota, f)$, where $(S^\sharp, \iota)$ is an untilt and $f \in X(A^{\sharp,+})$.
- "slashed diamond" $X^{\Diamond/} := X^{\diamond} \coprod_{X^{\diamond} \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p} (X_{\mathbb{Q}_p})^{\Diamond}$.

If X is proper over $\mathbb{Z}_p$, the three notions above are the same by the valuative criterion of properness.

For a pre-adic space X over $\mathbb{Z}_p$, the similarly-defined functor $X^{\Diamond}$ sending $S \in \mathrm{Ob}\,\mathrm{Perf}$ to $(S^\sharp, \iota, f : S^\sharp \to X)$ is also a v-sheaf over Perf by [SW20, Lem. 18.1.1].

1.1.3. *Reduction.* Recall the reduction functor introduced in [Gle25]. Let $\widetilde{\mathrm{SchPerf}}$ denote the topos on $\mathrm{PCAlg}^{\mathrm{op}}$, endowed with the v-topology. Let $\widetilde{\mathrm{Perf}}$ denote the topos on Perf equipped with the v-topology. Then $(\diamond, \mathrm{red})$ form a pair of adjoint functors between the topoi $(\widetilde{\mathrm{SchPerf}}, \widetilde{\mathrm{Perf}})$. Given $\mathrm{Spec}\,A \in \mathrm{SchPerf}$ and a v-sheaf $\mathcal{F}$, we denote $\mathcal{F}^{\mathrm{red}}(A) = \mathrm{Hom}(\mathrm{Spd}\,A, \mathcal{F})$. The reduction functor red commutes with finite limits.

Let $\mathcal{F}$ be a v-sheaf over $\mathrm{Spd}\,\mathbb{Z}_p$, let $\mathcal{F}_{\mathbb{F}_p} := \mathcal{F}_{\mathrm{Spd}\,\mathbb{F}_p} \to \mathrm{Spd}\,\mathbb{F}_p$ be the base change of $\mathcal{F} \to \mathrm{Spd}\,\mathbb{Z}_p$. We say that $\mathcal{F}_{\mathbb{F}_p}$ is the special fiber of $\mathcal{F}$. Note that $(\mathrm{Spd}\,\mathbb{Z}_p)^{\mathrm{red}}$ is represented by $\mathrm{Spec}\,\mathbb{F}_p$. We say that $\mathcal{F}$ is formally p-adic (in the sense of [Gle25, Def 3.20]), if $\mathcal{F}_{\mathrm{Spd}\,\mathbb{F}_p} = (\mathcal{F}^{\mathrm{red}})^{\diamond}$.

Recall the Witt affine Grassmannian $\mathrm{Gr}_{\mathcal{P}}^W$ defined in [Zhu17].

Definition 1.5. *The Witt affine Grassmannian $\mathrm{Gr}_{\mathcal{P}}^W$ is the functor defined over $\mathrm{PCAlg}^{\mathrm{op}}$, sending $\mathrm{Spec}\,R \in \mathrm{PCAlg}^{\mathrm{op}}$ to the set of $\mathcal{P}$-torsors on $\mathrm{Spec}\,W(R)$ together with a trivialization over $\mathrm{Spec}\,W(R)[1/p]$.*

It follows from definition that $\mathrm{Gr}_{\mathcal{P},\mathbb{F}_p} = (\mathrm{Gr}_{\mathcal{P}}^W)^{\diamond}$, $\mathrm{Gr}_{\mathcal{P}}^W$ is the reduction of $\mathrm{Gr}_{\mathcal{P}}$. $\mathrm{Gr}_{\mathcal{P}}^W$ is the étale sheafification of $L_{\mathbb{F}_p}\mathcal{P}/L_{\mathbb{F}_p}^+\mathcal{P}$, where $L_{\mathbb{F}_p}\mathcal{P}$ and $L_{\mathbb{F}_p}^+\mathcal{P}$ are the Witt loop group functor and the Witt positive loop group functor respectively. By [BS17, Cor. 9.6], $\mathrm{Gr}_{\mathcal{P}}^W$ is represented by an increasing union of perfections of quasi-projective schemes over $\mathbb{F}_p$ along closed immersions. Here we only need $\mathrm{GL}_n / \mathcal{P}$ to be quasi-affine for some faithful embedding $\mathcal{P} \to \mathrm{GL}_n$ over $\mathbb{Z}_p$.

1.1.4. *Over C. From now on,* let P be a linear algebraic group over $\mathbb{Q}_p$ with a Levi decomposition $P = U \rtimes G$, where $U = R_u P$ and G is the reductive quotient. Let $\mathcal{G}$ be a smooth affine model of G over $\mathbb{Z}_p$, and let $\mathcal{U}$ be a smooth affine model of U over $\mathbb{Z}_p$ such that the conjugation action of $\mathcal{G}(\breve{\mathbb{Z}}_p)$ stabilizes $\mathcal{U}(\breve{\mathbb{Z}}_p)$. By [BT84, 1.7.3 and Prop. 1.7.6], we can associate a smooth affine model $\mathcal{P} := \mathcal{U} \rtimes \mathcal{G}$ to P over $\mathbb{Z}_p$. Throughout this article, we work with such $\mathcal{P}$.

Let C be a perfectoid field of characteristic 0. We write the base changes of $\mathrm{Gr}_{\mathcal{P}}$ and $\mathrm{Gr}_{\mathcal{G}}$ as $\mathrm{Gr}_{P,C}$ and $\mathrm{Gr}_{G,C}$, respectively. Let $\mu : \mathbb{G}_{\mathrm{m}} \to P$ be a cocharacter over $\overline{\mathbb{Q}}_p$. We define $\mathrm{Gr}_{P,\mu} \subset \mathrm{Gr}_{P,C}$ as the L^+P-orbit of $\mu : L\mathbb{G}_{\mathrm{m}} \to LP$, i.e., the v-sheaf-theoretic image of $L^+P \times \mu \to \mathrm{Gr}_{P,C}$, and define $\mathrm{Gr}_{P,\leq\mu}$ as the v-sheaf-theoretic closure of $\mathrm{Gr}_{P,\mu}$ in $\mathrm{Gr}_{P,C}$.

Remark 1.6. *Recall that* $\mathrm{Gr}_{G,\mu}$ *(resp.* $\mathrm{Gr}_{G,\leq\mu}$*) has two definitions: one uses such orbital image (resp. and its closure) as in* [AGLR22, §4]*, and the other uses the pointwise boundedness condition as in* [SW20, §19]*. These two definitions coincide; see* [AGLR22, Cor. 4.6].

Lemma 1.7. *Fix a section* $G \to P$ *such that* μ *lifts to* G*. We then have a closed embedding* $\mathrm{Gr}_{G,C} \to \mathrm{Gr}_{P,C}$*, and* $\mathrm{Gr}_{P,(\leq)\mu} \subset \mathrm{Gr}_{P,C}$ *is the* L^+P*-orbit of* $\mathrm{Gr}_{G,(\leq)\mu} \subset \mathrm{Gr}_{G,C}$*, i.e., the* v*-sheaf-theoretic image of* $L^+P \times \mathrm{Gr}_{G,(\leq)\mu} \to \mathrm{Gr}_{P,C}$*. As a result,* $\mathrm{Gr}_{P,\mu}$ *(resp.* $\mathrm{Gr}_{P,\leq\mu}$*) agrees with the subfunctor of all maps* $S \to \mathrm{Gr}_{P,C}$ *such that, for every geometric point* $x : \mathrm{Spa}(D, D^+) \to S$*, we have* $x \in P(B^+_{\mathrm{dR}}(D^\sharp))\xi^\mu P(B^+_{\mathrm{dR}}(D^\sharp))/P(B^+_{\mathrm{dR}}(D^\sharp))$ *(resp.* $x \in P(B^+_{\mathrm{dR}}(D^\sharp))\xi^{\mu'} P(B^+_{\mathrm{dR}}(D^\sharp))/P(B^+_{\mathrm{dR}}(D^\sharp))$ *for some* $\mu' \leq \mu$*). Here we identify* $B_{\mathrm{dR}}(D^\sharp) \cong D^\sharp((\xi))$*.*

Proof. This follows mainly from Remark 1.6. Only need to show that the image of $L^+P \times \overline{\mathrm{Gr}_{G,\mu}}$ in $\mathrm{Gr}_{P,C}$ is $\overline{\mathrm{Gr}_{P,\mu}}$, here $\overline{(*)}$ means the closure under the v-topology. Since $\mathrm{Gr}_{G,\leq\mu}$ is proper, thus quasi-compact and quasi-separated, by [SW20, Prop. 20.2.3], the action $a : L^+P \times \mathrm{Gr}_{G,\leq\mu} \to \mathrm{Gr}_{P,C}$ factors through $a_N : L^{\leq N}P \times \mathrm{Gr}_{G,\leq\mu} \to \mathrm{Gr}_{P,C}$ for some large N and a_N is proper (see Lemma 1.4). We then apply Lemma 1.8. ☐

Lemma 1.8. *Let* $f : X_1 \to X_2$ *be a universally closed morphism of small* v*-stacks, and let* Y_1 *and* Y_2 *be sub-*v*-stacks of* X_1 *and* X_2*, respectively. Assume that* $f|_{Y_1}$ *factors through* Y_2 *and* $f : Y_1 \to Y_2$ *is surjective. Then* $f : Y_1^{\mathrm{cl}} \to Y_2^{\mathrm{cl}}$ *is surjective, where* Y_i^{cl} *is the* v*-closure of* Y_i *in* X_i*.*

Proof. By [AGLR22, Prop. 2.8], $Y_i^{\mathrm{cl}} = |Y_i|^{wgc} \times_{|Y_i|} X_i$, where $|Y_i|^{wgc}$ is the weakly generalizing closure of $|Y_i| \subset |X_i|$. Since $Y_1 \to Y_2$ is surjective, $|Y_1| \to |Y_2|$ is surjective (see [Sch26, Prop. 12.9]). By [AGLR22, Lem. 2.4], since f is universally closed, $|Y_2|^{wgc} \subset |f|(|Y_1|^{wgc})$, and $f : Y_1^{\mathrm{cl}} \to Y_2^{\mathrm{cl}}$ is surjective. ☐

Proposition 1.9 ([SW20, Prop. 19.2.3])**.** *Fix a maximal torus and a Borel* $T \subset B \subset G$*. Fix a dominant cocharacter* $\mu \in X_*(T)^+$*. Then* $\mathrm{Gr}_{P,\leq\mu} \subset \mathrm{Gr}_{P,C}$ *is a closed subfunctor, and* $\mathrm{Gr}_{P,\mu} \subset \mathrm{Gr}_{P,\leq\mu}$ *is an open subfunctor. Moreover,* $\mathrm{Gr}_{P,\leq\mu}$ *is a locally spatial diamond.*

Proof. Let $\mathrm{Gr}'_{P,\leq\mu} \subset \mathrm{Gr}_{P,C}$ be the pullback of $\mathrm{Gr}_{G,\leq\mu} \subset \mathrm{Gr}_{G,C}$ under the projection $\mathrm{Gr}_{P,C} \to \mathrm{Gr}_{G,C}$. Then we have $f : \mathrm{Gr}_{P,\leq\mu} \xrightarrow{\subset} \mathrm{Gr}'_{P,\leq\mu}$. Since $\mathrm{Gr}_{G,\leq\mu} \subset \mathrm{Gr}_{G,C}$ is closed by [SW20, Prop. 19.2.3], it suffices to show that f is closed.

We take a strictly totally disconnected perfectoid cover X of the small v-sheaf $\mathrm{Gr}_{G,\leq\mu}$ and claim that f_X is closed. Note that closedness of a morphism descends along a v-cover ([Sch26, Prop. 10.11]). Take a v-cover $\bigsqcup_{\mu'\leq\mu} X_{\mu'} \to X$, where $X_{\mu'}$ is a strictly totally disconnected perfectoid space that is a v-cover of $X \times_{\mathrm{Gr}_{G,\leq\mu}} \mathrm{Gr}_{G,\mu'}$. Fix a point $x \in X_{\mu'}$, and let $U = \mathrm{Spd}(R, R^+) \to X_{\mu'}$ be an étale neighborhood of x such that $\mathrm{Gr}_{G,C}(U) = G(B_{\mathrm{dR}}(R^\sharp))/G(B^+_{\mathrm{dR}}(R^\sharp))$. Since every étale covering of $X_{\mu'}$ splits, U is again a strictly totally disconnected space. Since the v-sheaves involved are partially proper, for simplicity we work with rank-1 points and assume $R^+ = R^\circ$. Set $\widetilde{\mathcal{O}} = B^+_{\mathrm{dR}}(R^\sharp)$ and $\widetilde{F} = B_{\mathrm{dR}}(R^\sharp)$. By Lemma 1.10 below, the fiber f_U is isomorphic to

$$U(\widetilde{\mathcal{O}})/U(\widetilde{\mathcal{O}}) \cap \xi^{\mu'}U(\widetilde{\mathcal{O}})\xi^{-\mu'} \hookrightarrow U(\widetilde{F})/\xi^{\mu'}U(\widetilde{\mathcal{O}})\xi^{-\mu'},$$

It is a closed embedding since $U(\widetilde{\mathcal{O}}) \hookrightarrow U(\widetilde{F})$ is a closed embedding by Lemma 1.3.

To prove the last statement, we take a faithful representation $\rho : P \to \mathrm{GL}_n$ such that GL_n/P is quasi-affine. Then $\mathrm{Gr}_P \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a locally closed embedding. Since $\mathrm{Gr}_{P,\leq\mu} \subset \mathrm{Gr}_P$ and $\mathrm{Gr}_{\mathrm{GL}_n,\leq\rho(\mu)} \subset \mathrm{Gr}_{\mathrm{GL}_n}$ are closed, it follows that $\mathrm{Gr}_{P,\leq\mu} \to \mathrm{Gr}_{\mathrm{GL}_n,\leq\rho(\mu)}$ is locally closed. Since

$\mathrm{Gr}_{\mathrm{GL}_n,\leq\rho(\mu)}$ is a spatial diamond by [SW20, Thm. 19.2.4], we conclude that $\mathrm{Gr}_{P,\leq\mu}$ is a locally spatial diamond (see [Sch26, Cor. 11.28]). ☐

Lemma 1.10. *Let $S = \mathrm{Spa}(R, R^\circ)$ be a strictly totally disconnected perfectoid space. Let x be an S-point of $\mathrm{Gr}_{G,\mu} \to \mathrm{Spd}(C, \mathcal{O}_C)$, associated with the untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp\circ})$. Up to replacing S with an open subspace, we have*

$$x = g\xi^\mu G(B^+_{\mathrm{dR}}(R^\sharp))/G(B^+_{\mathrm{dR}}(R^\sharp))$$

for some $g \in G(B^+_{\mathrm{dR}}(R^\sharp))$.

Proof. Let $L^+G(\mu) := \ker(L^+G \to \mathrm{Gr}_{G,\mu})$ be the stabilizer subgroup. Then $L^+G \to \mathrm{Gr}_{G,\mu}$ is an $L^+G(\mu)$-torsor. It suffices to show that the orbit map $L^+G \to \mathrm{Gr}_{G,\mu}$ has a section over S after replacing S with an open subspace. Consider the reduction map $\mathrm{red} : L^+G \to G$, $g \mapsto \bar{g}$, modulo ξ. Let $L^1G(\mu)$ be the kernel of the restriction of red to $L^+G(\mu)$, and let $G_\mu \subset G$ be the image of $L^+G(\mu)$. We have an exact sequence $1 \to L^1G(\mu) \to L^+G(\mu) \to G_\mu \to 1$. To show that $H^1_v(S, L^+G(\mu)) = 0$ after replacing S with an open subspace, it suffices to show that $H^1_v(S, L^1G(\mu)) = 0$ and $H^1_v(S, G_\mu) = 0$ after replacing S with an open subspace.

(1) $H^1_v(S, G_\mu) = 0$: Note that G_μ is a parabolic subgroup of G determined by μ; in particular, it is smooth. A G_μ-torsor on a perfectoid space S for the v-topology is a torsor for the étale topology. A general result for rigid groups can be found in [Heu26]. Since every étale covering of S splits, after replacing S with an open subspace, we have $H^1_v(S, G_\mu) = 0$.

(2) $H^1_v(S, L^1G(\mu)) = 0$: Recall that G is split over C, and $L^1G(\mu)$ is a subgroup of the pro-unipotent kernel $\ker(L^+G \to G)$. More precisely, $L^1G(\mu)$ admits a descending filtration $L^\bullet G(\mu)$ such that $L^nG(\mu)/L^{n+1}G(\mu)$ is isomorphic to some vector group $\mathfrak{g}^{(n)} \otimes \xi^n/\xi^{n+1}$, where $\mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \cdots \mathfrak{g} = \mathrm{Lie}\, G$ is an increasing filtration. Since $H^1_v(S, B^+_{\mathrm{dR}}/(\xi)) = H^1_v(S, \mathcal{O}_S) = 0$, and the transition morphisms in the inverse system of vector groups $\{L^1G(\mu)/L^nG(\mu)\}_n$ are surjective, we conclude that $H^1_v(S, L^1G(\mu)) = 0$. ☐

1.1.5. *Over $\mathbb{Q}_p$.* Let $\{\mu\}$ be the conjugacy class of μ in P, and E be the defining field of $\{\mu\}$. Note that the defining field of $\{\mu\}$ in P is the same as the one of $\{\mu\}$ in G, since we can fix a section $G \to P$ such that μ lifts to G. Since $\mathrm{Gr}_{G,\mu}$ and $\mathrm{Gr}_{G,\leq\mu}$ are descended to $\mathrm{Spd}\, E$ (which we again denote by $\mathrm{Gr}_{G,\mu}$ and $\mathrm{Gr}_{G,\leq\mu}$ respectively), $\mathrm{Gr}_{P,\mu}$ and $\mathrm{Gr}_{P,\leq\mu}$ are descended to $\mathrm{Spd}\, E$ (which we again denote by $\mathrm{Gr}_{P,\mu}$ and $\mathrm{Gr}_{P,\leq\mu}$ respectively), by the descriptions in Lemma 1.7. Moreover, $\mathrm{Gr}_{P,\mu} \hookrightarrow \mathrm{Gr}_{P,\leq\mu}$ is open and $\mathrm{Gr}_{P,\leq\mu} \subset \mathrm{Gr}_{P,E}$ is closed. There is a section $\mathrm{Gr}_{G,\leq\mu} \to \mathrm{Gr}_{P,\leq\mu}$ to the projection $\mathrm{Gr}_{P,\leq\mu} \to \mathrm{Gr}_{G,\leq\mu}$.

1.1.6. *Represented by rigid analytic spaces.* Let

$$P_\mu := \{g \in P \mid \lim_{t\to\infty} \mu(t)g\mu(t)^{-1} \textit{ exists}\}, \quad G_\mu := \{g \in G \mid \lim_{t\to\infty} \mu(t)g\mu(t)^{-1} \textit{ exists}\}.$$

Let $\mathcal{F}l_{P,\mu} := P/P_\mu$ and $\mathcal{F}l_{G,\mu} = G/G_\mu$. Note that $\mathcal{F}l_{G,\mu}$ is the usual flag variety. The space $\mathcal{F}l_{P,\mu}$ can be also viewed as a smooth quasi-projective scheme over E.

Definition 1.11.

(1) *We say μ is **quotient-minuscule** in P if μ is minuscule in the reductive quotient G.*
(2) *We say μ is **minuscule** in P if the weights of μ on $\mathrm{Lie}\, P$ are $\{-1, 0, 1\}$.*

It follows from the definition that μ is minuscule implies that μ is quotient-minuscule.

Lemma 1.12 ([SW20, Prop. 19.4.2])**.** *There is a Bialynicki-Birula map $\mathrm{Gr}_{P,\mu} \to \mathcal{F}l^\diamondsuit_{P,\mu}$, which is an isomorphism when μ is minuscule in P.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_{P,\mu} \arrow[r] \arrow[d] & \mathcal{F}l^{\lozenge}_{P,\mu} \arrow[d] \\
\mathrm{Gr}_{G,\mu} \arrow[r] & \mathcal{F}l^{\lozenge}_{G,\mu},
\end{tikzcd}$$

where the horizontal maps are defined using the Tannakian formalism and [SW20, Prop. 19.4.2], and the bottom map is an isomorphism. Since $\mathcal{F}l^{\lozenge}_{P,\mu}$ and $\mathrm{Gr}_{P,\mu}$ are diamonds, to check the qcqs map $\mathrm{Gr}_{P,\mu} \to \mathcal{F}l^{\lozenge}_{P,\mu}$ is an isomorphism, it suffices to check the bijectivity on geometric points by [Sch26, Lem. 11.11, Prop. 11.15]. In particular, we check the bijectivity of the fibers at every geometric point of $\mathrm{Gr}_{G,\mu} \xrightarrow{\sim} \mathcal{F}l^{\lozenge}_{G,\mu}$. Let $\mathrm{Spa}(C, C^+)$ be a geometric point of characteristic 0, denote $\mathcal{O} := B^+_{\mathrm{dR}}(C) \cong C[[\xi]]$ and $F = B_{\mathrm{dR}}(C) \cong C((\xi))$. We have identifications:

$$\mathrm{Gr}_{P,\mu}(C, C^+) = P(\mathcal{O})\xi^{\mu}P(\mathcal{O})/P(\mathcal{O}) = P(\mathcal{O})/P(\mathcal{O}) \cap \xi^{\mu}P(\mathcal{O})\xi^{-\mu}.$$

The natural morphism $\mathrm{Gr}_{P,\mu}(C, C^+) \to \mathcal{F}l^{\lozenge}_{P,\mu}(C, C^+)$ is induced by the reduction map

$$P(\mathcal{O}) \to P(C), \quad q \mapsto \bar{q} \mod \xi.$$

Let $g \in G(C)$, the fiber of $\mathrm{Gr}_{P,\mu} \to \mathrm{Gr}_{G,\mu}$ at g (more explicity, over $g\xi^{\mu}G(\mathcal{O})$) is

$$U(\mathcal{O})/(U(\mathcal{O}) \cap g\xi^{\mu}U(\mathcal{O})\xi^{-\mu}g^{-1}) \cong U(\mathcal{O})/(U(\mathcal{O}) \cap \xi^{\mu}U(\mathcal{O})\xi^{-\mu}). \tag{1.1}$$

and the fiber of $\mathcal{F}l^{\lozenge}_{P,\mu} \to \mathcal{F}l^{\lozenge}_{G,\mu}$ at g is

$$U(C)/gU_{\mu}(C)g^{-1} \cong U(C)/U_{\mu}(C),$$

where $\mathrm{Lie}\, U_{\mu}$ consists of μ-weight spaces of $\mathrm{Lie}\, U$ of weights ≤ 0. By the minuscule condition on μ, the reduction map gives the bijection

$$U(\mathcal{O})/(U(\mathcal{O}) \cap \xi^{\mu}U(\mathcal{O})\xi^{-\mu}) \xrightarrow{\sim} U(C)/U_{\mu}(C).$$

□

Definition 1.13. *We say that (P, μ)* ***comes from the boundary*** *if there exists a reductive $\mathbb{Q}_p$-group G' with a parabolic $\mathbb{Q}_p$-subgroup $Q \subset G'$ such that*

(1) $\mu : \mathbb{G}_{\mathrm{m}} \xrightarrow{\mu} P \to G'$ is a minuscule cocharacter in G' defined over a field extension of E,
(2) $\{U_{\alpha} | \ \langle \mu, \alpha \rangle > 0, \alpha \in \Phi'\} \subset Q$, where Φ' is the absolute root datum of G'.
(3) P is a normal subgroup of Q with $R_u P = R_u Q$.

In this case, we denote the Levi quotient of Q by L.

Lemma 1.14. *Let (G', X') be a pure Shimura datum and let (P, D) be a rational boundary component. Let $x' \in D$, and let $\mu_{x'}$ be the Hodge cocharacter associated with x'. Then $(P_{\mathbb{Q}_p}, \mu_{x', \overline{\mathbb{Q}}_p})$ comes from the boundary in the sense of Definition 1.13.*

Proof. The groups involved are defined over $\mathbb{Q}$, we work over the base field $\mathbb{C}$ in place of $\overline{\mathbb{Q}}_p$. Let H_0 be the reference group defined in [Pin90, §4.3]. It is defined over $\mathbb{R}$ and equipped with two homomorphisms $h_0 : \mathbb{S} \to H_0$ and $h_{\infty} : \mathbb{S}_{\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} \times H_{0,\mathbb{C}}$. Let Q be an admissible parabolic subgroup of G'. For any $x = h_x \in X'$ that determines a point $x' = h_{x'} \in D$ in the sense of [Pin90, §4.11], one can find a morphism from the reference group, $\omega_x : H_0 \to G'_{\mathbb{R}}$, such that $h_x = \omega_x \circ h_0 : \mathbb{S} \to G'_{\mathbb{R}}$ and $h_{x'} = \omega_x \circ h_{\infty} : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$; see [Pin90, Prop. 4.6].

Let $w : \mathbb{G}_{\mathrm{m}} \to \mathbb{S}$ and $\mu : (\mathbb{G}_{\mathrm{m}})_{\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ be the weight and Hodge homomorphisms, respectively. Let $\mu_x = h_x \circ \mu$ and $\mu_{x'} = h_{x'} \circ \mu$ be the Hodge cocharacters associated with $x \in X'$ and $x' \in D$, respectively. Then μ_x and $\mu_{x'}$ are conjugate under G' (indeed, they are already conjugate in H_0); see [Pin90, Prop. 12.1, Lem. 12.2]. In particular, $\mu_{x'}$ is minuscule.

Since the Hodge structure on $\operatorname{Lie} G'$ induced by h_x is pure of type $\{(-1,1),(0,0),(1,-1)\}$, [Pin90, Lem. 4.4] implies that the Hodge structure induced by $h_{x'}$ is mixed, with types

$$\{(1,1),(1,0),(0,1),(1,-1),(0,0),(-1,1),(0,-1),(-1,0),(-1,-1)\}. \tag{1.2}$$

The adjoint action of $\mu_{x'}:\mathbb{G}_{\mathrm{m}}\to P_{\mathbb{C}}\to G'_{\mathbb{C}}$ on $\operatorname{Lie} U_\alpha$ is given by multiplication by $t^{\langle\mu_{x'},\alpha\rangle}$. Thus $\operatorname{Lie} U_\alpha\subset F^0\operatorname{Lie} G'$ (F^0 is the Hodge filtration induced by $h_{x'}$) if and only if $\langle\mu_{x'},\alpha\rangle\le 0$ (see conventions in [Pin90, §1.3]). In particular, if $\langle\mu_{x'},\alpha\rangle>0$, then the weight of $\operatorname{Lie} U_\alpha$ under $h_{x'}\circ w$ has to be at least 0 by (1.2). Hence $\operatorname{Lie} U_\alpha\subset(\operatorname{Lie} Q)_{\mathbb{C}}$, since $(\operatorname{Lie} Q)_{\mathbb{C}}$ is the direct sum of all nonnegative weight spaces in $(\operatorname{Lie} G')_{\mathbb{C}}$ under the adjoint action of $h_{x'}\circ w$; see [Pin90, 4.1 and Prop. 4.6]. □

Lemma 1.15. *If (P,μ) comes from the boundary, then the immersions $\mathcal{F}l_{P,\mu}\to\mathcal{F}l_{Q,\mu}\to\mathcal{F}l_{G',\mu}$ are open immersions; in particular, they have the same dimension.*

Proof. This follows from a calculation of the tangent spaces at the identity. Since the construction of $\mathcal{F}l$ is compatible with base change, we base change everything to an algebraically closed field. We take a maximal torus T inside Q,

$$\operatorname{Lie} G/\operatorname{Lie} G_\mu=\oplus_{\alpha\in\Phi,\langle\mu,\alpha\rangle>0}\operatorname{Lie} U_\alpha=\operatorname{Lie} Q/\operatorname{Lie} Q_\mu.$$

Let us denote $W:=R_uP=R_uQ$, $G_h:=P/W$, $L:=Q/W$, and $G_l:=Q/P$. We have an exact sequence of reductive groups $1\to G_h\to L\to G_l\to 1$, and the root data of G_h and G_l are orthogonal. In particular, $\operatorname{Lie} L/\operatorname{Lie} L_\mu=\operatorname{Lie} G_h/\operatorname{Lie} G_{h,\mu}$; thus $\operatorname{Lie} Q/\operatorname{Lie} Q_\mu=\operatorname{Lie} P/\operatorname{Lie} P_\mu$. □

Remark 1.16. *Under the setting of Lemma 1.14, let us give a more geometric explanation of Lemma 1.15. Consider the map $\tau:X\to\pi_0(X)\times\operatorname{Hom}(\mathbb{S}_{\mathbb{C}},P_{\mathbb{C}})$ that sends x to $([x],\omega_x\circ h_\infty)$. Let (P,D) be a rational boundary component. Recall that $D\subset\pi_0(X)\times\operatorname{Hom}(\mathbb{S}_{\mathbb{C}},P_{\mathbb{C}})$ is a $P(\mathbb{R})U(\mathbb{C})$-orbit containing $\tau(x)$ for some $x\in X$. Let $\phi:D\to\phi(D)\subset\operatorname{Hom}(\mathbb{S}_{\mathbb{C}},P_{\mathbb{C}})$ be the natural projection with finite fibers. Fix $h\in\phi(D)$, let $C(h)$ be the centralizer of h in $P(\mathbb{R})W(\mathbb{C})$. Let M be a faithful representation of P. By the proof of* [Pin90, Prop. 1.7]*, we have*

$$\phi(D)=P(\mathbb{R})W(\mathbb{C})/C(h)\hookrightarrow P(\mathbb{C})/\exp(F^0(\operatorname{Lie} P)_{\mathbb{C}})=P(\mathbb{C})/P_\mu(\mathbb{C})\hookrightarrow\operatorname{Grass}(M)(\mathbb{C}),$$

where the first morphism is an open embedding, and the last morphism is the closed embedding given by the Hodge filtration associated with the Hodge cocharacter μ determined by h. In particular, the dimension of D agrees with that of $\mathcal{F}l_{P,\mu}$ as complex manifolds. This is analogous to the Borel embedding $X'\hookrightarrow\tilde{X}'=G'/G'_{\mu'}=\mathcal{F}l_{G',\mu'}$, which also shows that the dimension of X' equals that of $\mathcal{F}l_{G',\mu'}$. Let $X'\subset X$ be the preimage of D under τ, which is a union of connected components of X. Then $X'\to D$ is an open embedding; see [Pin90, Prop. 4.15(a)]*. Thus X' and D have the same dimension. Finally, since μ and μ' are conjugate in G', we conclude that $\mathcal{F}l_{P,\mu}$ and $\mathcal{F}l_{G',\mu}$ have the same dimension.*

Lemma 1.17. *If (P,μ) comes from the boundary, then $\operatorname{Gr}_{P,\mu}\to\mathcal{F}l^{\Diamond}_{P,\mu}$ and $\operatorname{Gr}_{Q,\mu}\to\mathcal{F}l^{\Diamond}_{Q,\mu}$ are isomorphisms. In particular, $\operatorname{Gr}_{P,\mu}$ and $\operatorname{Gr}_{Q,\mu}$ are represented by rigid analytic spaces over* $\operatorname{Spd} E$*. Moreover, $\operatorname{Gr}_{P,\mu}\xrightarrow{\sim}\operatorname{Gr}_{Q,\mu}$ and $\mathcal{F}l_{P,\mu}\xrightarrow{\sim}\mathcal{F}l_{Q,\mu}$.*

Proof. The first two statements follow from Lemma 1.12. Let us show the last statement. Given any $q\in Q(\mathcal{O})$, we can factor $q=p\prod_\alpha u_\alpha$ with $p\in P(\mathcal{O})$ and $u_\alpha\in U_\alpha(\mathcal{O})$, where α runs over roots in $Q\setminus P$. Since μ factors through P, we have $\langle\alpha,\mu\rangle=0$ for all such α. In particular, $U_\alpha(\mathcal{O})\subset Q(\mathcal{O})\cap\xi^\mu Q(\mathcal{O})\xi^{-\mu}$, and hence $\operatorname{Gr}_{P,\mu}(C,C^+)=\operatorname{Gr}_{Q,\mu}(C,C^+)$, $\operatorname{Gr}_{P,\mu}\xrightarrow{\sim}\operatorname{Gr}_{Q,\mu}$. Since $\mathcal{F}l_{P,\mu}$ and $\mathcal{F}l_{Q,\mu}$ are smooth, and the $\Diamond$ functor is fully faithful on seminormal rigid analytic spaces ([SW20, Prop. 10.2.3]), $\mathcal{F}l_{P,\mu}\xrightarrow{\sim}\mathcal{F}l_{Q,\mu}$. □

1.1.7. *Over* $\operatorname{Spd}\mathbb{Z}_p$. We move on to the integral base and recall the definition of v-sheaf theoretical local models.

Definition 1.18. *We say that $\mathcal{P}$ is a* ***stabilizer quasi-parahoric group scheme*** *(resp.* ***quasi-parahoric group scheme, parahoric group scheme****) of P if $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$, $\mathcal{G}$ is a stabilizer quasi-parahoric group scheme (resp. quasi-parahoric group scheme, parahoric group scheme) of G and $\mathcal{U}$ is a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers. In these cases, denote $\mathcal{P}^\circ = \mathcal{U} \rtimes \mathcal{G}^\circ$.*

Let $Q \subset G'$ be a parabolic subgroup, and let $T \subset Q$ be a maximal $\breve{\mathbb{Q}}_p$-split torus of G' defined over $\mathbb{Q}_p$. Let $U = R_u Q$, and let L be the standard Levi subgroup of Q containing T.

Let $\mathcal{F} \subset A(G',T)$ be a facet in the reduced apartment. Under the natural projection $A(G',T) \to A(L,T)$, there is a unique facet $\mathcal{F}_L \subset A(L,T)$ containing $\mathcal{F}$. Let $\mathcal{L}_{\mathcal{F}_L}$ (resp. $\mathcal{G}'_{\mathcal{F}}$) be the parahoric group scheme of L (resp. G') with respect to $\mathcal{F}_L$ (resp. $\mathcal{F}$). Then $\mathcal{L}_{\mathcal{F}_L}(\breve{\mathbb{Z}}_p) = L(\breve{\mathbb{Q}}_p) \cap \mathcal{G}'_{\mathcal{F}}(\breve{\mathbb{Z}}_p)$; see [HR10, Lem. 4.1.1]. Moreover, $\mathcal{L}_{\mathcal{F}_L} \to \mathcal{G}'_{\mathcal{F}}$ is a closed embedding. To see this, let $\mathcal{L}_{\mathcal{F}}$ be the closure of L in $\mathcal{G}'_{\mathcal{F}}$. Then $\mathcal{L}_{\mathcal{F}_L} \to \mathcal{L}_{\mathcal{F}}$ is the smoothing; i.e., $\mathcal{L}_{\mathcal{F}_L}$ is the unique smooth affine group scheme of L such that $\mathcal{L}_{\mathcal{F}_L}(\breve{\mathbb{Z}}_p) = \mathcal{L}_{\mathcal{F}}(\breve{\mathbb{Z}}_p)$. Since $\mathcal{L}_{\mathcal{F}}$ contains the open big cell of $\mathcal{L}_{\mathcal{F}_L}$, we have $\mathcal{L}_{\mathcal{F}_L} \xrightarrow{\sim} \mathcal{L}_{\mathcal{F}}$. Let $\mathcal{U}_{\mathcal{F}}$ be the closure of U in $\mathcal{G}'_{\mathcal{F}}$; it is a smooth affine group scheme with connected fibers, since its special fiber is a product of root groups U_α inside the linear algebraic group $\mathcal{G}'_{\mathcal{F},\mathbb{F}_p}$. The closure of Q in $\mathcal{G}'_{\mathcal{F}}$ is $\mathcal{Q}_{\mathcal{F}} := \mathcal{U}_{\mathcal{F}} \rtimes \mathcal{L}_{\mathcal{F}}$, and $\mathcal{Q}_{\mathcal{F}}$ is parahoric.

Definition 1.19. *We say that an embedding of parahoric group schemes $\mathcal{Q} \subset \mathcal{G}'$ is a* parabolic embedding *if there exists a facet $\mathcal{F} \subset A(G',T)$ for some maximal $\breve{\mathbb{Q}}_p$-split torus T of G' contained in Q such that $\mathcal{Q} = \mathcal{Q}_{\mathcal{F}}$ and $\mathcal{G}' = \mathcal{G}'_{\mathcal{F}}$.*

Let $\mathrm{Gr}_{\mathcal{G}',\leq\mu}$ be the closure of $\mathrm{Gr}_{G',\leq\mu}$ in $\mathrm{Gr}_{\mathcal{G}'}$; it is proper and is represented by a spatial diamond (see [SW20, Prop. 20.3.6]).

Lemma 1.20. *Let $\mathcal{Q} \subset \mathcal{G}'$ be a parabolic embedding of parahoric group schemes, and let μ be a cocharacter of G' factoring through Q. Then $\mathrm{Gr}_{\mathcal{Q},\leq\mu} \to \mathrm{Gr}_{\mathcal{G}',\leq\mu}$ is a locally closed embedding, where $\mathrm{Gr}_{\mathcal{Q},\leq\mu}$ is the closure of the closed subfunctor $\mathrm{Gr}_{Q,\leq\mu} \subset \mathrm{Gr}_Q$ (see Proposition 1.9) in $\mathrm{Gr}_{\mathcal{Q},\mathcal{O}_E}$, and E is the defining field of $\{\mu\}$.*

Proof. Denote G' by G. We apply the arguments in [AGLR22, §5.1]. Let $T \subset Q \subset G$ be a maximal $\breve{\mathbb{Q}}_p$-split torus defined over $\mathbb{Q}_p$ such that $\mathcal{F} \subset A(G,T)$. Let $\mathcal{T}^\circ$ be the connected Néron model of T, and let $\lambda : \mathbb{G}_{\mathrm{m}} \to \mathcal{T}^\circ$ be a cocharacter defined over $\mathbb{Z}_p$ such that $Q = G_\lambda$ and $L = G_\lambda \cap G_{\lambda^{-1}}$. As in the proof of [HR21, Lem. 4.5], λ defines $\mathcal{Q} = \mathcal{G}_\lambda$ and $\mathcal{L} = \mathcal{G}_\lambda \cap \mathcal{G}_{\lambda^{-1}}$. Choose an embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,\mathbb{Z}_p}$ such that $\mathrm{GL}_{n,\mathbb{Z}_p}/\mathcal{G}$ is quasi-affine. Let $T' \subset G' := \mathrm{GL}_n$ be a maximal $\breve{\mathbb{Q}}_p$-split torus defined over $\mathbb{Q}_p$ that contains T, and let $Q' := G'_{\lambda'}$, where λ' is the composition of $\mathbb{G}_{\mathrm{m}} \xrightarrow{\lambda} \mathcal{T}^\circ \to \mathcal{T}'^\circ$. Then $Q' = U' \rtimes L'$ extends to $\mathcal{Q}' = \mathcal{U}' \rtimes \mathcal{L}'$ where $\mathcal{Q}'$ is again a parahoric group scheme. As in the arguments in the proof of [AGLR22, Thm. 5.2], $\mathcal{Q}'/\mathcal{Q}$ is quasi-affine; thus $\mathrm{Gr}_{\mathcal{Q}} \to \mathrm{Gr}_{\mathcal{Q}'}$ is a locally closed immersion. By the discussion above Lemma 1.20, it suffices to work with $\mathcal{Q}' \subset \mathcal{G}'$. To save notation, we denote by $(\mathcal{Q}', \mathcal{G}', \mu') = (\mathcal{Q}, \mathcal{G}, \mu)$. Now $\mathcal{Q} = \mathcal{U} \rtimes \mathcal{L}$, and $\mathcal{L}$ and $\mathcal{G}$ are reductive schemes over $\mathbb{Z}_p$ since T is a split maximal torus. Let T_L be the cocenter of L, and let ν be the image of μ under $L \to T_L$. We use results from [FS21, Prop. III.3.6] (the proof there works over the integral base, since [FS21, Lem. VI.3.2, VI.3.3] do.) Recall that we have a locally constant Kottwitz map $\mathrm{Gr}_{\mathcal{L}} \to \pi_1(L)$. Let $\mathrm{Gr}_{\mathcal{L}}^\nu \subset \mathrm{Gr}_{\mathcal{L}}$ be the corresponding open and closed sub-sheaf, and let $\mathrm{Gr}_{\mathcal{Q}}^\nu \subset \mathrm{Gr}_{\mathcal{Q}}$ be the preimage of $\mathrm{Gr}_{\mathcal{L}}^\nu \subset \mathrm{Gr}_{\mathcal{L}}$. By [FS21, Prop. VI.3.1], $\mathrm{Gr}_{\mathcal{Q}}^\nu \subset \mathrm{Gr}_{\mathcal{G}}$ is locally closed. Since $\mathrm{Gr}_{\mathcal{L},\leq\mu} \subset \mathrm{Gr}_{\mathcal{L}}^\nu$, $\mathrm{Gr}_{\mathcal{Q},\leq\mu} \subset \mathrm{Gr}_{\mathcal{Q}}$ and $\mathrm{Gr}_{\mathcal{G},\leq\mu} \subset \mathrm{Gr}_{\mathcal{G}}$ are closed embeddings, $\mathrm{Gr}_{\mathcal{Q},\leq\mu} \to \mathrm{Gr}_{\mathcal{G},\leq\mu}$ is locally closed. □

Let $G_h \subset L$ be a normal subgroup with quotient G_l. Then G_h and G_l are reductive. Moreover, L is an almost product of G_h and G'_l, where $G'_l \to G_l$ is an isogeny. Recall that we have a

$G_h(\breve{\mathbb{Q}}_p) \times G'_l(\breve{\mathbb{Q}}_p)$-$\sigma$-equivariant bijection between reduced Bruhat–Tits buildings that preserves the polysimplicial structure:

$$\mathcal{B}_{\mathrm{red}}(G_h, \breve{\mathbb{Q}}_p) \times \mathcal{B}_{\mathrm{red}}(G_l, \breve{\mathbb{Q}}_p) \to \mathcal{B}_{\mathrm{red}}(L, \breve{\mathbb{Q}}_p), \quad (\mathcal{F}_h, \mathcal{F}_l) \leftrightarrow \mathcal{F}_L, \quad (x_h, x_l) \leftrightarrow x_L. \tag{1.3}$$

In particular, $\mathcal{G}_{h,x_h}(\breve{\mathbb{Z}}_p) = G_h(\breve{\mathbb{Q}}_p) \cap \mathcal{L}_{x_L}(\breve{\mathbb{Z}}_p)$. However, the parahoric group schemes $\mathcal{G}_{h,\mathcal{F}_h}$ and $\mathcal{L}_{\mathcal{F}_L}$ might not satisfy $\mathcal{G}_{h,\mathcal{F}_h}(\breve{\mathbb{Z}}_p) = G_h(\breve{\mathbb{Q}}_p) \cap \mathcal{L}_{\mathcal{F}_L}(\breve{\mathbb{Z}}_p)$; see [Mao25a, §2.4.3] (especially [Mao25a, Example 2.61]) for detailed discussions. Moreover, the closure of G_h in $\mathcal{L}_{x_L}$ might not be smooth. It is smooth if there exists a maximal $\breve{\mathbb{Q}}_p$-split torus of G_h that is R-smooth; see [Mao25a, Lem. 2.77].

We make the following definition, which will be used when considering mixed Shimura varieties at the boundary.

Definition 1.21. *Assume that (P, μ) comes from the boundary, with $P \to Q \to G'$ as usual. Let $\mathcal{P} \to \mathcal{Q} \to \mathcal{G}'$ be quasi-parahoric group schemes of $P \to Q \to G'$. We say that $(\mathcal{P}, \mu)$ comes from the boundary associated with $\mathcal{P} \to \mathcal{Q} \to \mathcal{G}'$ if*

1. *$\mathcal{Q}$ (resp. $\mathcal{P}$) is the smoothing of the closure of Q (resp. P) in $\mathcal{G}'$.*
2. *$\mathcal{P}^\circ \to \mathcal{Q}^\circ \to \mathcal{G}'^\circ$ is of the form $\mathcal{P}_{\mathcal{F}_h} \to \mathcal{Q}_{\mathcal{F}_L} \to \mathcal{G}'_{\mathcal{F}}$, with facets $\mathcal{F}_h$, $\mathcal{F}_L$, $\mathcal{F}$ related in the way discussed above.*

1.1.8. *Pass to parahoric levels.* In the remainder of this section, we always assume that $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$ is a quasi-parahoric group scheme.

Lemma 1.22.

$$\begin{tikzcd}
\mathrm{Gr}_{\mathcal{P}^\circ, \operatorname{Spd} \mathbb{Z}_p} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{P}, \operatorname{Spd} \mathbb{Z}_p} \arrow[d] \\
\mathrm{Gr}_{\mathcal{G}^\circ, \operatorname{Spd} \mathbb{Z}_p} \arrow[r] & \mathrm{Gr}_{\mathcal{G}, \operatorname{Spd} \mathbb{Z}_p},
\end{tikzcd} \qquad \begin{tikzcd}
\mathrm{Gr}^W_{\mathcal{P}^\circ} \arrow[r] \arrow[d] & \mathrm{Gr}^W_{\mathcal{P}} \arrow[d] \\
\mathrm{Gr}^W_{\mathcal{G}^\circ} \arrow[r] & \mathrm{Gr}^W_{\mathcal{G}}.
\end{tikzcd}$$

are Cartesian diagrams.

Proof. The commutativity of both diagrams follows from the fact that the sheafification functor commutes with fiber products, and the diagram of presheaves is clearly Cartesian (since $\mathcal{P} \to \mathcal{G}$ is surjective). ☐

Lemma 1.23. *The fibers of $\mathrm{Gr}^W_{\mathcal{P}} \to \mathrm{Gr}^W_{\mathcal{G}}$ are connected. Moreover, $\mathrm{Gr}^W_{\mathcal{P}} \to \mathrm{Gr}^W_{\mathcal{G}}$ induces a bijection $\pi_0(\mathrm{Gr}^W_{\mathcal{P}}) \to \pi_0(\mathrm{Gr}^W_{\mathcal{G}})$.*

Proof. Recall that $L\mathcal{U} = LU$ is connected as an ind-scheme by fixing some $U \cong \mathbb{A}^n$, and $L^+\mathcal{U}$ is also connected when $\mathcal{U}$ is connected. Hence, $\mathrm{Gr}^W_{\mathcal{U}}$ is connected as an ind-scheme. Let $x = \operatorname{Spec} l \to \mathrm{Gr}^W_{\mathcal{G}}$ be a geometric point, and write $x = g\mathcal{G}(W(l))/\mathcal{G}(W(l))$. Then the fiber of $\mathrm{Gr}^W_{\mathcal{P}} \to \mathrm{Gr}^W_{\mathcal{G}}$ at x is $U(W(l)[1/p])/g\mathcal{U}(W(l)[1/p])g^{-1}$, which is conjugate to $\mathrm{Gr}^W_{\mathcal{U}}$. Moreover, fixing a section $\mathcal{G} \to \mathcal{P}$, since the action $LU \times \mathrm{Gr}^W_{\mathcal{G}} \to \mathrm{Gr}^W_{\mathcal{P}}$ is surjective, $\mathrm{Gr}^W_{\mathcal{P}} \to \mathrm{Gr}^W_{\mathcal{G}}$ induces a bijection $\pi_0(\mathrm{Gr}^W_{\mathcal{P}}) \to \pi_0(\mathrm{Gr}^W_{\mathcal{G}})$. ☐

Corollary 1.24 ([Zhu17, Prop. 1.21])**.** *$\mathrm{Gr}^W_{\mathcal{P}^\circ} \to \mathrm{Gr}^W_{\mathcal{P}}$ is a $\mathcal{P}/\mathcal{P}^\circ$-torsor that maps connected components isomorphically onto connected components.*

Proof. This statement is true when $\mathcal{P} = \mathcal{G}$; see [Zhu17, Prop. 1.21] (cf. [SW20, Prop. 21.1.4]). In general, $\mathrm{Gr}^W_{\mathcal{P}^\circ} \to \mathrm{Gr}^W_{\mathcal{P}}$ is a $\mathcal{P}/\mathcal{P}^\circ$-torsor since $\mathrm{Gr}^W_{\mathcal{G}^\circ} \to \mathrm{Gr}^W_{\mathcal{G}}$ is a $\mathcal{G}/\mathcal{G}^\circ$-torsor and $\mathcal{P}/\mathcal{P}^\circ = \mathcal{G}/\mathcal{G}^\circ$ by Lemma 1.22. The last statement then follows from Lemma 1.23. ☐

1.2. **Local models.** In the remainder of this section, we always assume that μ is **quotient-minuscule** in the sense of Definition 1.11. We define

$$\mathbb{M}^v_{\mathcal{P},\mu} \subset \mathrm{Gr}_{\mathcal{P},\mathrm{Spd}\,\mathcal{O}_E} = \mathrm{Gr}_{\mathcal{P},\mathrm{Spd}\,\mathbb{Z}_p} \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathcal{O}_E \tag{1.4}$$

as the closure of $\mathrm{Gr}_{P,\mu}$. The closed embedding $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{P}}$ induces $\mathbb{M}^v_{\mathcal{G},\mu} \to \mathbb{M}^v_{\mathcal{P},\mu}$, which gives a section of the projection $\mathbb{M}^v_{\mathcal{P},\mu} \to \mathbb{M}^v_{\mathcal{G},\mu}$, thus $\mathbb{M}^v_{\mathcal{P},\mu}$ projects onto $\mathbb{M}^v_{\mathcal{G},\mu}$.

Lemma 1.25 ([SW20, Prop. 21.4.3])**.** *The natural morphism $\mathbb{M}^v_{\mathcal{P}^\circ,\mu} \to \mathbb{M}^v_{\mathcal{P},\mu}$ is an isomorphism.*

Proof. This statement is true when $\mathcal{P} = \mathcal{G}$; see [SW20, Prop. 21.4.3]. In general, we use Lemma 1.22. Since $\mathrm{Gr}_{\mathcal{G}^\circ,\mathrm{Spd}\,\mathbb{Z}_p} \to \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\,\mathbb{Z}_p}$ is proper, $\mathrm{Gr}_{\mathcal{P}^\circ,\mathrm{Spd}\,\mathbb{Z}_p} \to \mathrm{Gr}_{\mathcal{P},\mathrm{Spd}\,\mathbb{Z}_p}$ is also proper. In particular, $\mathbb{M}^v_{\mathcal{P}^\circ,\mu} \to \mathbb{M}^v_{\mathcal{P},\mu}$ is proper. Applying the arguments in the proof of [SW20, Prop. 21.4.3], and using Corollary 1.24, it suffices to show that the reduction of $\mathbb{M}^v_{\mathcal{P}^\circ,\mu}$ (see Lemma 1.29) is completely contained in one connected component of $\mathrm{Gr}^W_{\mathcal{P}^\circ}$. Again, this follows from Corollary 1.24 and the fact that $\mathbb{M}^v_{\mathcal{P}^\circ,\mu}$ projects onto $\mathbb{M}^v_{\mathcal{G}^\circ,\mu}$. ☐

Proposition 1.26. *Assume $(\mathcal{P},\mu)$ comes from the boundary with $\mathcal{P} \to \mathcal{Q} \to \mathcal{G}'$ (see Definition 1.21); then the natural embedding $\mathbb{M}^v_{\mathcal{P},\mu} \xrightarrow{\sim} \mathbb{M}^v_{\mathcal{Q},\mu} \to \mathbb{M}^v_{\mathcal{G}',\mu}$ is a locally closed embedding.*

Proof. Lemma 1.17 says that $\mathrm{Gr}_{P,\mu} \to \mathrm{Gr}_{Q,\mu}$ is an isomorphism. Since $\mathcal{Q}^\circ/\mathcal{P}^\circ$ is affine, $\mathrm{Gr}_{\mathcal{P}^\circ} \to \mathrm{Gr}_{\mathcal{Q}^\circ}$ is a closed embedding, see Lemma 1.2. In particular, $\mathbb{M}^v_{\mathcal{P}^\circ,\mu} \xrightarrow{\sim} \mathbb{M}^v_{\mathcal{Q}^\circ,\mu}$. By Lemma 1.25, we have $\mathbb{M}^v_{\mathcal{P},\mu} \xrightarrow{\sim} \mathbb{M}^v_{\mathcal{Q},\mu}$. By Lemma 1.20 and Lemma 1.25, $\mathbb{M}^v_{\mathcal{Q},\mu} \to \mathbb{M}^v_{\mathcal{G}',\mu}$ is a locally closed embedding. ☐

1.2.1. *Orbital descriptions.*

Lemma 1.27 ([AGLR22, Prop. 4.13])**.** *The local model $\mathbb{M}^v_{\mathcal{P},\mu}$ is stable under $L^+\mathcal{P}$-action. Moreover, $\mathbb{M}^v_{\mathcal{P},\mu}$ is the $L^+\mathcal{P}$-orbit of $\mathbb{M}^v_{\mathcal{G},\mu}$ under the fixed closed embedding $\mathrm{Gr}_{\mathcal{G}} \subset \mathrm{Gr}_{\mathcal{P}}$.*

Proof. Choosing a faithful embedding $\rho : \mathcal{P} \to \mathrm{GL}_n$ such that $\mathrm{GL}_n/\mathcal{P}$ is quasi-affine, we have a locally closed embedding $\mathrm{Gr}_{\mathcal{P}} \to \mathrm{Gr}_{\mathrm{GL}_n}$. Let $X \subset \mathrm{Gr}_{\mathrm{GL}_n,\mathcal{O}_E}$ be a quasi-compact and quasi-separated closed subsheaf that contains $\mathbb{M}^v_{\mathcal{P},\mu}$ (for example, take $X = \mathrm{Gr}_{\mathrm{GL}_n,\le\rho(\mu),\mathcal{O}_E}$). Since $\mathbb{M}^v_{\mathcal{P},\mu} \to X$ is a monomorphism, as in Lemma 1.4, $a : L^+_{\mathcal{O}_E}\mathcal{P}\times\mathbb{M}^v_{\mathcal{P},\mu} \to \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E}$ factors through $a_N : L^{\le N}_{\mathcal{O}_E}\mathcal{P}\times\mathbb{M}^v_{\mathcal{P},\mu} \to \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E}$ for some large N, where N is determined by X. Let $X_1 = X \cap \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E} \hookrightarrow \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E}$, it is a closed sub-sheaf, $L^+_{\mathcal{O}_E}\mathcal{P}$ acts on X_1 and factors through $L^{\le N}_{\mathcal{O}_E}\mathcal{P}$. Since $L^{\le N}_{\mathcal{O}_E}\mathcal{P} \to \mathrm{Spd}\,\mathcal{O}_E$ is partially proper and open, by [AGLR22, Cor. 2.9] (which says that taking closure commutes with base change along partially proper and open morphisms), base change along the projection $L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} X_1 \to X_1$, we have

$$L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathbb{M}^v_{\mathcal{P},\mu} = (L^{\le N}_{E}\mathcal{P} \times_{\mathrm{Spd}\,E} \mathrm{Gr}_{P,\mu})^{\mathrm{cl}},$$

as a closed subsheaf of $L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} X_1$. Since $L^{\le N}_{E}\mathcal{P} \times_{\mathrm{Spd}\,E} \mathrm{Gr}_{P,\mu} \to X_{1,E}$ factors through $\mathrm{Gr}_{P,\mu}$ by construction, combining with the equality displayed, we see that $L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathbb{M}^v_{\mathcal{P},\mu} \to X_1$ factors through $\mathbb{M}^v_{\mathcal{P},\mu}$.

Let us show the second statement. Since $\mathbb{M}^v_{\mathcal{G},\mu} \subset \mathbb{M}^v_{\mathcal{P},\mu}$, $\mathbb{M}^v_{\mathcal{P},\mu}$ is stable under $L^+\mathcal{P}$-action implies that the morphism $a' : L^+_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathbb{M}^v_{\mathcal{G},\mu} \to \mathrm{Gr}_{\mathcal{P},\mathrm{Spd}\,\mathcal{O}_E}$ factors through $\mathbb{M}^v_{\mathcal{P},\mu}$. We need to show it is surjective. Since $\mathbb{M}^v_{\mathcal{G},\mu}$ is quasi-compact and quasi-separated, apply Lemma 1.4, a' factors through some a'_N for large N. Apply [AGLR22, Cor. 2.9] again. Note that $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{P,\mu}$, as well as $\mathbb{M}^v_{\mathcal{G},\mu} \subset \mathbb{M}^v_{\mathcal{P},\mu}$, is a closed embedding. Base-changing along the projection $L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathbb{M}^v_{\mathcal{P},\mu} \to \mathbb{M}^v_{\mathcal{P},\mu}$, we have

$$L^{\le N}_{\mathcal{O}_E}\mathcal{P} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathbb{M}^v_{\mathcal{G},\mu} = (L^{\le N}_{E}\mathcal{P} \times_{\mathrm{Spd}\,E} \mathrm{Gr}_{G,\mu})^{\mathrm{cl}},$$

as a closed subsheaf of $L^{\leq N}_{\mathcal{O}_E}\mathcal{P} \times_{\operatorname{Spd}\mathcal{O}_E} \mathbb{M}^v_{\mathcal{P},\mu}$. Since $L^{\leq N}_{\mathcal{O}_E}\mathcal{P} \times_{\operatorname{Spd}E} \mathrm{Gr}_{G,\mu}$ surjects onto $\mathrm{Gr}_{P,\mu}$ by construction, we apply Lemma 1.8 to the action a'_N. $\square$

1.2.2. *Over* $\operatorname{Spd}\mathbb{F}_p$. Let $\mathcal{G}$ be parahoric. Let $M^{\mathrm{loc}}_{\mathcal{G},\mu} \subset \mathrm{Gr}^W_{\mathcal{G},k_E}$ be the reduction of $\mathbb{M}^v_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$. We have $M^{\mathrm{loc}}_{\mathcal{G},\mu} = \mathcal{A}_{\mathcal{G},\mu} \subset \mathrm{Gr}^W_{\mathcal{G},k_E}$ (see [AGLR22, Thm. 1.5]). Here $\mathcal{A}_{\mathcal{G},\mu}$ is the μ-admissible locus in the sense of Kottwitz–Rapoport, which is the k_E descent of

$$\mathcal{A}_{\mathcal{G},\mu,k} = \bigcup_{w \in \mathrm{Adm}_G(\{\mu^{-1}\})_{\mathcal{G}(\breve{\mathbb{Z}}_p)}} \mathrm{Gr}^W_{\mathcal{G},w,k},$$

where $k = \bar{k}_E$, $\mathrm{Gr}^W_{\mathcal{G},w,k}$ is the usual Schubert cell, it is the perfection of a quasi-projective scheme.

Let $\mathcal{P}$ be parahoric and $\widetilde{W}$ be the Iwahori-Weyl group of G. For any $w \in \widetilde{W}$, let $\mathrm{Gr}^W_{\mathcal{P},w} \subset \mathrm{Gr}^W_{\mathcal{P},k}$ be the orbit of $L^+_k\mathcal{P}$ of $\mathrm{Gr}^W_{\mathcal{G},w} \to \mathrm{Gr}^W_{\mathcal{G},k} \to \mathrm{Gr}^W_{\mathcal{P},k}$. By calculating the stabilizer of the orbit of $L^+_k\mathcal{P}$ at w, we see that $\mathrm{Gr}^W_{\mathcal{P},w}$ is locally perfectly of finite type. Let us define the μ-admissible locus $\mathcal{A}_{\mathcal{P},\mu} \subset \mathrm{Gr}^W_{\mathcal{P},k_E}$ as the k_E-descent of

$$\mathcal{A}_{\mathcal{P},\mu,k} = \bigcup_{w \in \mathrm{Adm}_P(\{\mu^{-1}\})_{\mathcal{P}(\breve{\mathbb{Z}}_p)}} \mathrm{Gr}^W_{\mathcal{P},w}, \quad \mathrm{Adm}_P(\{\mu^{-1}\})_{\mathcal{P}(\breve{\mathbb{Z}}_p)} := \mathrm{Adm}_G(\{\mu^{-1}\})_{\mathcal{G}(\breve{\mathbb{Z}}_p)}. \tag{1.5}$$

Let $\breve{K} = \mathcal{G}(\mathbb{Z}_p)$, and let ${}_{\breve{K}}\widetilde{W}^{\breve{K}} \subset \widetilde{W}$ be the index set introduced in [Ric13] (cf. [SYZ21, §1.2.6]).

Lemma 1.28. *Given* $w \in {}_{\breve{K}}\widetilde{W}^{\breve{K}}$, *let* $\overline{\mathrm{Gr}^W_{\mathcal{P},w}}$ *be the closure of* $\mathrm{Gr}^W_{\mathcal{P},w} \subset \mathrm{Gr}^W_{\mathcal{P},k}$, *then*

$$\overline{\mathrm{Gr}^W_{\mathcal{P},w}} = \mathrm{Gr}^W_{\mathcal{P},\leq w} := \bigcup_{w' \leq w, w' \in {}_{\breve{K}}\widetilde{W}^{\breve{K}}} \mathrm{Gr}^W_{\mathcal{P},w'}\,.$$

In particular, $\mathcal{A}_{\mathcal{P},\mu} \subset \mathrm{Gr}^W_{\mathcal{P},k_E}$ *is a closed subfunctor.*

Proof. Since the closure of $\mathrm{Gr}^W_{\mathcal{G},w}$ in $\mathrm{Gr}^W_{\mathcal{G}}$ is $\mathrm{Gr}^W_{\mathcal{G},\leq w}$, the usual affine Schubert variety. Use the orbit map $L^+_k\mathcal{P} \times \mathrm{Gr}^W_{\mathcal{G},k} \to \mathrm{Gr}^W_{\mathcal{P},k}$, we see that $\mathrm{Gr}^W_{\mathcal{P},\leq w} \subset \overline{\mathrm{Gr}^W_{\mathcal{P},w}}$. On the other hand, by Lemma 1.4, the action $a : L^+_k\mathcal{P} \times \mathrm{Gr}^W_{\mathcal{G},\leq w} \to \mathrm{Gr}^W_{\mathcal{P},k}$ factors through a_N for some large N, and a_N is proper. In particular, $L^+_k\mathcal{P} \times \mathrm{Gr}^W_{\mathcal{G},\leq w}$ has closed image in $\mathrm{Gr}_{\mathcal{P}}$ that contains $\mathrm{Gr}_{\mathcal{P},w}$, thus $\overline{\mathrm{Gr}^W_{\mathcal{P},w}} \subset \mathrm{Gr}^W_{\mathcal{P},\leq w}$. $\square$

Lemma 1.29. *Let* $M^{\mathrm{loc}}_{\mathcal{P},\mu} \subset \mathrm{Gr}^W_{\mathcal{P},k_E}$ *be the reduction of* $\mathbb{M}^v_{\mathcal{P},\mu} \subset \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E}$. *Then* $M^{\mathrm{loc}}_{\mathcal{P},\mu} \subset \mathrm{Gr}^W_{\mathcal{P},k_E}$ *is representable and closed.*

Proof. We need to show that $\mathbb{M}^v_{\mathcal{P},\mu}$ is ϖ-adic formal (i.e. the structure morphism to $\operatorname{Spd}\mathcal{O}_E$ is formally adic in the sense of [Gle25, Def. 3.20]); then $M^{\mathrm{loc}}_{\mathcal{P},\mu} \subset \mathrm{Gr}^W_{\mathcal{P},k_E}$ represents the base change of $\mathbb{M}^v_{\mathcal{P},\mu} \subset \mathrm{Gr}_{\mathcal{P},\mathcal{O}_E}$ along $\operatorname{Spd}\mathbb{F}_q \to \operatorname{Spd}\mathcal{O}_E$. To show the claim, we follow the proof in [AGLR22, Prop. 4.14]: We need to show that the special fiber of $\mathbb{M}^v_{\mathcal{P},\mu}$ is represented by a perfect scheme. This follows from Lemma 1.27 and in particular (1.5). $\square$

By checking the set of k-points of these two closed functors, we have

Corollary 1.30. *Let* $\mathcal{P}$ *be parahoric. Then* $M^{\mathrm{loc}}_{\mathcal{P},\mu} = \mathcal{A}_{\mathcal{P},\mu}$.

Remark 1.31. *Recall that,* $\mathbb{M}^v_{\mathcal{G},\mu}$ *is a proper topologically normal rich kimberlite, due to results in* [Gle25], [AGLR22], [GL24]. *One might expect that* $\mathbb{M}^v_{\mathcal{P},\mu}$ *is also a topologically normal rich kimberlite, though not necessarily proper. Most arguments in the cited papers work here, except for one crucial ingredient, the purity of* $\mathcal{G}$*-torsor proved in* [Ans22]. *This purity result simply fails for non-reductive group, in particular,* $\mathrm{Gr}_{\mathcal{P}}$ *might not be v-formalizing, thus might not be pre-kimberlite.*

Let $\mathcal{G}_0$ be the special fiber of $\mathcal{G}$. The $L^+_{k_E}\mathcal{G}$-action on $M^{\mathrm{loc}}_{\mathcal{G},\mu}$ factors through $\mathcal{G}_0$ (see [PR24, §4.9.1]). In particular, we have $|\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}| = \mathrm{Adm}_G(\{\mu^{-1}\})_{\mathcal{G}(\breve{\mathbb{Z}}_p)}$. Similarly, let $\mathcal{P}_0$ be the special fiber of $\mathcal{P}$. Assume $(\mathcal{P},\mu)$ comes from the boundary, associated with $\mathcal{P}\to\mathcal{Q}\to\mathcal{G}'$. Use the monomorphisms $\mathcal{P}\to\mathcal{G}'$ and $\mathbb{M}^v_{\mathcal{P},\mu}\to\mathbb{M}^v_{\mathcal{G}',\mu}$, we have

Corollary 1.32. *Assume $(\mathcal{P},\mu)$ comes from the boundary. Then the $L^+_{k_E}\mathcal{P}$-action on $M^{\mathrm{loc}}_{\mathcal{P},\mu}$ factors through $\mathcal{P}_0$. In particular, $|\mathcal{P}_0\backslash M^{\mathrm{loc}}_{\mathcal{P},\mu}| = \mathrm{Adm}_P(\{\mu^{-1}\})_{\mathcal{P}(\breve{\mathbb{Z}}_p)}$.*

1.2.3. *Functoriality.*

Definition 1.33. *Let $P_1 = U_1\rtimes G_1$, $P_2 = U_2\rtimes G_2$. We say $P_1\to P_2$ is compatible with $U_1\rtimes G_1\to U_2\rtimes G_2$ if $P_1\to P_2$ induces $U_1\to U_2$, $G_1\to G_2$, and there exist sections $G_1\to P_1$ and $G_2\to P_2$ such that $G_1\to P_1\to P_2$ factors through G_2.*

Remark 1.34.

- *In this case, $P_1\to P_2$ is uniquely determined by the maps $U_1\to U_2$ and $G_1\to G_2$.*
- *When $P_1\to P_2$ induces $U_1\to U_2$, $G_1\to G_2$, then one can always find such a pair of sections $G_1\to P_1$ and $G_2\to P_2$ in characteristic 0.*

Let $P_1\to P_2$ be compatible with $U_1\rtimes G_1\to U_2\rtimes G_2$, and let $\mathcal{P}_1\to\mathcal{P}_2$ be a morphism of quasi-parahoric group schemes that extends $P_1\to P_2$; then $\mathcal{P}_1\to\mathcal{P}_2$ induces $\mathcal{U}_1\to\mathcal{U}_2$ and $\mathcal{G}_1\to\mathcal{G}_2$.

Definition 1.35. *We say $\mathcal{P}_1\to\mathcal{P}_2$ is compatible with $\mathcal{G}_1\to\mathcal{G}_2$ if there exist sections $\mathcal{G}_1\to\mathcal{P}_1$, $\mathcal{G}_2\to\mathcal{P}_2$ such that $\mathcal{G}_1\to\mathcal{P}_1\to\mathcal{P}_2$ factors through $\mathcal{G}_2$.*

In this subsection, we consider the following case. Let $P_1 = U\rtimes G_1\to P_2 = U\rtimes G_2$ be compatible with $U\rtimes G_1\overset{(\mathrm{id},f)}{\to} U\rtimes G_2$. Let $\mathcal{P}_1=\mathcal{U}\rtimes\mathcal{G}_1$ and $\mathcal{P}_2=\mathcal{U}\rtimes\mathcal{G}_2$ be parahoric group schemes of P_1 and P_2, respectively, such that $P_1\to P_2$ induces $\mathcal{P}_1\to\mathcal{P}_2$ compatible with $\mathcal{G}_1\to\mathcal{G}_2$.

Recall that the sheafification functor commutes with fiber products. Since $\mathcal{P}_2\to\mathcal{G}_2$ is surjective with a section $\mathcal{G}_2\to\mathcal{P}_2$, we have:

Lemma 1.36. *The following diagrams are Cartesian:*

$$\begin{tikzcd}
\mathrm{Gr}_{P_1} \arrow[r] \arrow[d] & \mathrm{Gr}_{P_2} \arrow[d] \\
\mathrm{Gr}_{G_1} \arrow[r] & \mathrm{Gr}_{G_2},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Gr}_{\mathcal{P}_1} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{P}_2} \arrow[d] \\
\mathrm{Gr}_{\mathcal{G}_1} \arrow[r] & \mathrm{Gr}_{\mathcal{G}_2},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Gr}^W_{\mathcal{P}_1} \arrow[r] \arrow[d] & \mathrm{Gr}^W_{\mathcal{P}_2} \arrow[d] \\
\mathrm{Gr}^W_{\mathcal{G}_1} \arrow[r] & \mathrm{Gr}^W_{\mathcal{G}_2}.
\end{tikzcd}$$

Assume that the cocharacter μ of P_1 is quotient-minuscule in both P_1 and P_2.

Lemma 1.37. *We have a Cartesian diagram:*

$$\begin{tikzcd}
\mathrm{Gr}_{P_1,\mu} \arrow[r] \arrow[d] & \mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1,\mu} \arrow[d] \\
\mathrm{Gr}_{G_1,\mu} \arrow[r] & \mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1,\mu}.
\end{tikzcd}$$

In particular, $\mathrm{Gr}_{P_1,\mu}\to\mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1,\mu}$ is proper.

Proof. For $i = 1, 2$, let $\mathrm{Gr}'_{P_i,\mu,\mathrm{Spd}\,E_1} \subset \mathrm{Gr}_{P_i,\mathrm{Spd}\,E_1}$ be the base change of $\mathrm{Gr}_{G_i,\mathrm{Spd}\,E_1,\mu} \subset \mathrm{Gr}_{G_i,\mathrm{Spd}\,E_1}$. Consider the commutative diagram with vertical diagrams being Cartesian:

$$\begin{tikzcd}
\mathrm{Gr}'_{P_1,\mu} \arrow[rrr] \arrow[dr] \arrow[dd] & & & \mathrm{Gr}'_{P_2,\mathrm{Spd}\,E_1,\mu} \arrow[dl] \arrow[dd] \\
& \mathrm{Gr}_{P_1,\mathrm{Spd}\,E_1} \arrow[r] \arrow[dd] & \mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1} \arrow[dd] & \\
\mathrm{Gr}_{G_1,\mu} \arrow[rrr] \arrow[dr] & & & \mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1,\mu} \arrow[dl] \\
& \mathrm{Gr}_{G_1,\mathrm{Spd}\,E_1} \arrow[r] & \mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1}. &
\end{tikzcd}$$

Both $\mathrm{Gr}_{G_1,\mu}$ and $\mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1,\mu}$ are proper over $\mathrm{Spd}\,E_1$, thus $\mathrm{Gr}_{G_1,\mu} \to \mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1,\mu}$ is proper. By Proposition 1.9, $\mathrm{Gr}_{P_i,\mathrm{Spd}\,E_1,\mu} \to \mathrm{Gr}'_{P_i,\mathrm{Spd}\,E_1,\mu}$ are closed embeddings for $i = 1, 2$. In particular, $\mathrm{Gr}_{P_1,\mu} \to \mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1,\mu}$ is proper. In order to show the proper morphism between diamonds

$$\mathrm{Gr}_{P_1,\mu} \to \mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1,\mu} \times_{\mathrm{Gr}_{G_2,\mathrm{Spd}\,E_1,\mu}} \mathrm{Gr}_{G_1,\mu} \tag{1.6}$$

is an isomorphism, it suffices to check it is an isomorphism over any $\mathrm{Spa}(C, C^+)$-point in $\mathrm{Gr}_{G_1,\mu}$. Let $\mathcal{O} = B^+_{\mathrm{dR}}(C^\sharp)$, $F = B_{\mathrm{dR}}(C^\sharp)$, $g\xi^\mu G(\mathcal{O}) \in \mathrm{Gr}_{G_1,\mu}(C, C^+)$, with $g \in G_1(\mathcal{O})$. Since g normalizes $U(\mathcal{O}) \rtimes G_i(\mathcal{O})$, then the fibers of both sides of (1.6) over $g\xi^\mu G(\mathcal{O})$ are given by the same set (1.1). $\square$

Lemma 1.38. *We have cartesian diagrams:*

$$\begin{tikzcd}
M^{\mathrm{loc}}_{\mathcal{P}_1,\mu} \arrow[r] \arrow[d] & M^{\mathrm{loc}}_{\mathcal{P}_2,\mu,k_{E_1}} \arrow[d] & & \mathbb{M}^v_{\mathcal{P}_1,\mu} \arrow[r] \arrow[d] & \mathbb{M}^v_{\mathcal{P}_2,\mu} \times_{\mathrm{Spd}\,\mathcal{O}_{E_2}} \mathrm{Spd}\,\mathcal{O}_{E_1} \arrow[d] \\
M^{\mathrm{loc}}_{\mathcal{G}_1,\mu} \arrow[r] & M^{\mathrm{loc}}_{\mathcal{G}_2,\mu,k_{E_1}}, & & \mathbb{M}^v_{\mathcal{G}_1,\mu} \arrow[r] & \mathbb{M}^v_{\mathcal{G}_2,\mu} \times_{\mathrm{Spd}\,\mathcal{O}_{E_2}} \mathrm{Spd}\,\mathcal{O}_{E_1}.
\end{tikzcd} \tag{1.7}$$

Proof. Apply similar arguments appeared in the proof of Lemma 1.37, $\mathbb{M}^v_{\mathcal{P}_1,\mu} \to \mathbb{M}^v_{\mathcal{P}_2,\mu} \times_{\mathrm{Spd}\,\mathcal{O}_{E_2}} \mathrm{Spd}\,\mathcal{O}_{E_1}$ is proper. In particular, to show the proper morphism

$$\mathbb{M}^v_{\mathcal{P}_1,\mu} \to \mathbb{M}^v_{\mathcal{P}_2,\mu,\mathrm{Spd}\,\mathcal{O}_{E_1}} \times_{\mathbb{M}^v_{\mathcal{G}_2,\mu,\mathrm{Spd}\,\mathcal{O}_{E_1}}} \mathbb{M}^v_{\mathcal{G}_1,\mu}$$

is an isomorphism, it suffices to check over geometric points (see [SW20, Cor. 17.4.8]). Over generic fiber, this is Lemma 1.37. Over special fiber, this is the proper morphism

$$M^{\mathrm{loc}}_{\mathcal{P}_1,\mu} \to M^{\mathrm{loc}}_{\mathcal{P}_2,\mu,k_{E_1}} \times_{M^{\mathrm{loc}}_{\mathcal{G}_2,\mu,k_{E_1}}} M^{\mathrm{loc}}_{\mathcal{G}_1,\mu}$$

induced by the left diagram in (1.7). We apply the same arguments as in the proof of Lemma 1.37 and compute the fibers of both sides over $M^{\mathrm{loc}}_{\mathcal{G}_1,\mu}$, with the help of Corollary 1.30 (cf. Lemma 1.27) and Lemma 1.25. $\square$

1.2.4. *Devissage.* Let $\widetilde{P} \to P$ be a surjective morphism with kernel a central multiplicative group $Z \subset \widetilde{P}$. Let $\widetilde{G}$ be the Levi quotient of $\widetilde{P}$. Note that Z maps isomorphically into a central multiplicative group in $\widetilde{G}$. Let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{U}} \rtimes \widetilde{\mathcal{G}}$ (resp. $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$) be a quasi-parahoric group scheme of $\widetilde{P}$ (resp. P). Assume $\widetilde{P} \to P$ induces $\widetilde{\mathcal{P}} \to \mathcal{P}$ and $\widetilde{\mathcal{U}} = \mathcal{U}$.

Lemma 1.39 ([SW20, Prop. 21.5.2])**.** *Let $\widetilde{\mu}$ be a quotient-minuscule cocharacter of $\widetilde{P}$ and μ be its projection in P defined over $E \subset \widetilde{E}$, then the natural morphisms are isomorphisms:*

$$\mathbb{M}^v_{\widetilde{\mathcal{G}},\widetilde{\mu}} \overset{\sim}{\to} \mathbb{M}^v_{\mathcal{G},\mu} \otimes_{\mathrm{Spd}\,\mathcal{O}_E} \mathcal{O}_{\widetilde{E}}, \quad \mathbb{M}^v_{\widetilde{\mathcal{P}},\widetilde{\mu}} \overset{\sim}{\to} \mathbb{M}^v_{\mathcal{P},\mu} \otimes_{\mathrm{Spd}\,\mathcal{O}_E} \mathcal{O}_{\widetilde{E}}.$$

Proof. See [SW20, Prop. 21.5.2], Lemma 1.38 and 1.25. $\square$

1.3. **Shtukas with one leg bounded by μ.** Let $\mathcal{P}$ be a quasi-parahoric group scheme, and $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to P_{\overline{\mathbb{Q}}_p}$ be a quotient-minuscule cocharacter.

Definition 1.40 ([PR24, Def. 2.4.3])**.** *Let S be a perfectoid space over $k = \overline{\mathbb{F}}_p$, and let $S^\sharp$ be an untilt of S over $\mathcal{O}_{\breve{E}}$. A $\mathcal{P}$-shtuka over S with one leg at $S^\sharp$ is a pair $(\mathscr{P}, \phi_{\mathscr{P}})$, where*

(1) $\mathscr{P}$ is a $\mathcal{P}$-torsor over the adic space $S \dot{\times} \mathbb{Z}_p$,
(2) $\phi_{\mathscr{P}}$ is a $\mathcal{P}$-torsor isomorphism

$$\phi_{\mathscr{P}} : \mathrm{Frob}_S^*(\mathscr{P})|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp} \xrightarrow{\sim} \mathscr{P}|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp},$$

which is meromorphic along the closed Cartier divisor $S^\sharp \subset S \dot{\times} \mathbb{Z}_p$.

We say $(\mathscr{P}, \phi_{\mathscr{P}})$ is bounded by μ if the relative position $\phi_{\mathscr{P}}(\mathrm{Frob}_S^*(\mathscr{P}))$ and $\mathscr{P}$ at $S^\sharp$ is bounded by $\mathbb{M}^v_{\mathcal{P},\mu}$ (1.4).

Definition 1.41 ([PR24, Def. 2.4.8])**.** *Given a v-sheaf $\mathcal{F}$, a $\mathcal{P}$-shtuka over $\mathcal{F}$ (resp. with one leg bounded by μ) is a section of the v-stack (*[SW20, Prop. 19.5.3]*) given by the groupoid of $\mathcal{P}$-shtukas with one leg (resp. with one leg bounded by μ) over $\mathcal{F}$.*

1.3.1. *Functoriality.* Let $P_1 \to P_2$ be a morphism, $\mathcal{P}_1$, $\mathcal{P}_2$ be quasi-parahoric models of P_1, P_2 respectively, such that $P_1 \to P_2$ induces a morphism $\mathcal{P}_1 \to \mathcal{P}_2$. Let μ_1 be a cocharacter of P_1 with reflex field E_1, and let μ_2 be the composition of μ_1 with $P_1 \to P_2$, with reflex field E_2. Assume μ_1 and μ_2 are quotient-minuscule in P_1 and P_2 respectively. Then the natural functor $\mathrm{Gr}_{\mathcal{P}_1} \to \mathrm{Gr}_{\mathcal{P}_2}$ induces

$$\mathrm{Gr}_{P_1,\mu_1} \to \mathrm{Gr}_{P_2,\mathrm{Spd}\, E_1,\mu_2}, \quad \mathbb{M}^v_{\mathcal{P}_1,\mu_1} \to \mathbb{M}^v_{\mathcal{P}_2,\mu_2} \times_{\mathrm{Spd}\, \mathcal{O}_{E_2}} \mathrm{Spd}\, \mathcal{O}_{E_1}.$$

We also have

$$\mathrm{Sht}_{\mathcal{P}_1} \to \mathrm{Sht}_{\mathcal{P}_2}, \quad \mathrm{Sht}_{\mathcal{P}_1,\mu_1} \to \mathrm{Sht}_{\mathcal{P}_2,\mu_2},$$

which come from pushing out the $\mathcal{P}_1$-torsor to a $\mathcal{P}_2$-torsor and naturally extending the isomorphism of $\mathcal{P}_1$-torsors $\phi_{\mathcal{P}_1}$ to an isomorphism of $\mathcal{P}_2$-torsors $\phi_{\mathcal{P}_2}$.

The projection $\mathrm{Sht}_{\mathcal{P},\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ has a section $\mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Sht}_{\mathcal{P},\mu}$. The substack $\mathrm{Sht}_{\mathcal{P},\mu} \subset \mathrm{Sht}_{\mathcal{P}}$ is contained in the preimage of $\mathrm{Sht}_{\mathcal{G},\mu} \subset \mathrm{Sht}_{\mathcal{G}}$, but they are not equal.

1.3.2. *Trivial cocharacter.*

Definition 1.42. *We say $(\mathscr{P}, \phi_{\mathscr{P}})$ is a trivial $\mathcal{P}$-shtuka over $\mathcal{F}$ if for each perfectoid space $S \to \mathcal{F}$, $\mathscr{P}_S$ is a trivial $\mathcal{P}$-torsor over $S \dot{\times} \mathbb{Z}_p$, and $\phi_{\mathscr{P}_S}$ is the isomorphism induced by the Frobenius on the base S.*

In the following, we assume that X is a normal scheme, flat and of finite type over $\mathcal{O}_E$. By [PR24, Cor. 2.7.10], a $\mathcal{P}$-shtuka over $X^{\lozenge/}$ is uniquely determined by its generic fiber over $X^{\lozenge}_\eta$. In particular, a $\mathcal{P}$-shtuka over $X^{\lozenge/}$ is trivial if it is trivial over $X^{\lozenge}_\eta$.

Let $1 \to P_1 \to P_2 \to P_3 \to 1$ be an exact sequence of linear algebraic groups which induces exact sequences

$$1 \to U_1 \to U_2 \to U_3 \to 1, \quad 1 \to G_1 \to G_2 \to G_3 \to 1.$$

Let $1 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_3 \to 1$ be an exact sequence of quasi-parahoric group schemes extending the generic fiber which induces

$$1 \to \mathcal{U}_1 \to \mathcal{U}_2 \to \mathcal{U}_3 \to 1, \quad 1 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 1.$$

In other words, in the decomposition $\mathcal{P}_i = \mathcal{U}_i \rtimes \mathcal{G}_i$, we assume there exist compatible sections $\mathcal{G}_i \to \mathcal{P}_i$. Let μ be a quotient-minuscule cocharacter of P_1, then it is automatically quotient-minuscule in P_2.

Proposition 1.43. *In the above setting, let* $(\mathscr{P}_2, \phi_{\mathscr{P}_2})$ *be a* $\mathcal{P}_2$*-shtuka over* $X^{\Diamond/}$ *(resp. with one leg bounded by* μ*). Assume that its base change over* $X^{\Diamond}_{\eta}$ *reduces to a* $\mathcal{P}_1$*-shtuka* $(\mathscr{P}_{1,\eta}, \phi_{\mathscr{P}_{1,\eta}})$ *over* $X^{\Diamond}_{\eta}$ *(resp. with one leg bounded by* μ*). Then* $(\mathscr{P}_2, \phi_{\mathscr{P}_2})$ *reduces uniquely to a* $\mathcal{P}_1$*-shtuka* $(\mathscr{P}_1, \phi_{\mathscr{P}_1})$ *(resp. with one leg bounded by* μ*) over* $X^{\Diamond/}$ *whose base change over* $X^{\Diamond}_{\eta}$ *is* $(\mathscr{P}_{1,\eta}, \phi_{\mathscr{P}_{1,\eta}})$*.*

Proof. Consider the pushout $X^{\Diamond/} \to \mathrm{Sht}_{\mathcal{P}_2} \to \mathrm{Sht}_{\mathcal{P}_3}$. The pushout $\mathcal{P}_3$-shtuka $(\mathscr{P}_3, \phi_{\mathscr{P}_3})$ is trivial over $X^{\Diamond}_{\eta}$, and thus is itself trivial over $X^{\Diamond/}$. In other words, given any perfectoid space $S \to X^{\Diamond/}$, the $\mathcal{P}_2$-torsor $\mathscr{P}_2$ over $S \dot{\times} \mathbb{Z}_p$ reduces to a $\mathcal{P}_1$-torsor $\mathscr{P}_1$, and the isomorphism

$$\Phi_{\mathscr{P}_{2,S}} : \mathrm{Frob}_S^* \mathscr{P}_{2,S}|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp} \cong \mathscr{P}_{2,S}|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp}$$

reduces to an isomorphism of $\mathcal{P}_1$-torsor:

$$\Phi_{\mathscr{P}_{1,S}} : \mathrm{Frob}_S^* \mathscr{P}_{1,S}|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp} \cong \mathscr{P}_{1,S}|_{S \dot{\times} \mathbb{Z}_p \backslash S^\sharp}.$$

Let us check the boundedness condition.

1. When P_3 is a reductive group, equivalently, we have that $U_1 = U_2$. By [SW20, Prop. 21.5.1], $\mathbb{M}^v_{\mathcal{G}_1,\mu} \xrightarrow{\sim} \mathbb{M}^v_{\mathcal{G}_2,\mu} \otimes \mathrm{Spd}\,\mathcal{O}_{E_1}$ is a canonical isomorphism. Lemma 1.38 gives $\mathbb{M}^v_{\mathcal{P}_1,\mu} \xrightarrow{\sim} \mathbb{M}^v_{\mathcal{P}_2,\mu} \otimes \mathrm{Spd}\,\mathcal{O}_{E_1}$. Hence, $(\mathscr{P}_1, \phi_{\mathscr{P}_1})$ is bounded by μ.
2. When P_3 is a unipotent group, equivalently, we have that $G_1 = G_2$, we claim that the following commutative diagram of v-sheaves is Cartesian. In particular, $(\mathscr{P}_1, \phi_{\mathscr{P}_1})$ is bounded by μ.

$$\begin{tikzcd}
\mathbb{M}^v_{\mathcal{P}_1,\mu} \arrow[r] \arrow[d] & \mathbb{M}^v_{\mathcal{P}_2,\mu} \times_{\mathrm{Spd}\,\mathcal{O}_{E_2}} \mathrm{Spd}\,\mathcal{O}_{E_1} \arrow[d] \\
\mathrm{Gr}_{\mathcal{P}_1,\mathrm{Spd}\,\mathcal{O}_{E_1}} \arrow[r] & \mathrm{Gr}_{\mathcal{P}_2,\mathrm{Spd}\,\mathcal{O}_{E_1}}.
\end{tikzcd}$$

The bottom morphism is a closed embedding since $\mathcal{P}_2/\mathcal{P}_1 = \mathcal{U}$ is affine; then the top morphism is a closed embedding by construction. Then the induced morphism

$$\mathbb{M}^v_{\mathcal{P}_1,\mu} \to \mathbb{M}^v_{\mathcal{P}_2,\mu,\mathrm{Spd}\,\mathcal{O}_{E_1}} \times_{\mathrm{Gr}_{\mathcal{P}_2,\mathrm{Spd}\,\mathcal{O}_{E_1}}} \mathrm{Gr}_{\mathcal{P}_1,\mathrm{Spd}\,\mathcal{O}_{E_1}} \tag{1.8}$$

is also closed. To show that it is an isomorphism, it suffices to check the surjectivity on fibers over the $\mathrm{Spa}(C, C^+)$-points in $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\,\mathcal{O}_{E_1}}$.

Over the generic fiber, (1.8) becomes

$$\mathrm{Gr}_{P_1,\mu} \to \mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1,\mu} \times_{\mathrm{Gr}_{P_2,\mathrm{Spd}\,E_1}} \mathrm{Gr}_{P_1,\mathrm{Spd}\,E_1}. \tag{1.9}$$

Let $\mathcal{O} = B^+_{\mathrm{dR}}(C^\sharp)$, $F = B_{\mathrm{dR}}(C^\sharp)$, fix a point $g\xi^\mu G(\mathcal{O}) \in \mathrm{Gr}_{G,\mu}(C, C^+)$, where $g \in G(\mathcal{O})$, let $h = g\xi^\mu$, the fiber of the right hand side of (1.9) over h is

$$U_2(\mathcal{O})/(U_2(\mathcal{O}) \cap \xi^\mu U_2(\mathcal{O})\xi^{-\mu}) \times_{U_2(F)/\xi^\mu U_2(\mathcal{O})\xi^{-\mu}} U_1(F)/\xi^\mu U_1(\mathcal{O})\xi^{-\mu},$$

as subset of $U_1(F)/\xi^\mu U_1(\mathcal{O})\xi^{-\mu}$, equals the fiber of the left hand side of (1.9) over h:

$$U_1(\mathcal{O})/(U_1(\mathcal{O}) \cap \xi^\mu U_1(\mathcal{O})\xi^{-\mu}),$$

since $U_1(F) \subset U_2(F)$ (resp. $U_1(\mathcal{O}_F) \subset U_2(\mathcal{O}_F)$) is stabilized by the conjugation of $P_1(F)$ (resp. $P_1(\mathcal{O}_F)$). Over the special fiber, the isomorphism can be checked similarly, with the help of Corollary 1.30 (cf. Lemma 1.27) and Lemma 1.25.
3. In general, we can insert an auxiliary $\mathcal{P}'$ into

$$\mathcal{P}_1 = \mathcal{U}_1 \rtimes \mathcal{G}_1 \to \mathcal{P}' = \mathcal{U}_2 \rtimes \mathcal{G}_1 \to \mathcal{P}_2 = \mathcal{U}_2 \rtimes \mathcal{G}_2,$$

where $\mathcal{G}_1$ acts on $\mathcal{U}_2$ through $\mathcal{G}_1 \to \mathcal{G}_2$, and apply the above two steps successively.

□

1.3.3. Bun_P *and* $P\text{-}\mathrm{Isoc}$. Recall that $P\text{-}\mathrm{Isoc}$ is a stack on $\mathrm{PCAlg}^{\mathrm{op}}$ that maps $S = \operatorname{Spec} R$ to the groupoid of pairings $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a P-torsor over $\operatorname{Spec} W(R)[1/p]$, and $\beta : \sigma^*\mathcal{E} \to \mathcal{E}$ is an isomorphism of P-torsors. By [GI23, Thm 7.14, Prop. 4.7, Prop. 6.3], $P\text{-}\mathrm{Isoc}$ is a v-stack, and $P\text{-}\mathrm{Isoc} \cong \mathrm{Bun}_P^{\mathrm{red}}$, the proof of this part does not need P to be reductive. By definition there is a bijection of the underlying topological spaces $|P\text{-}\mathrm{Isoc}| \cong B(P)$.

Let μ be a quotient-minuscule cocharacter of P, also denote by μ its projection on G. Recall that one defines $G\text{-}\mathrm{Isoc}_\mu \subset G\text{-}\mathrm{Isoc}$ as the closed substack corresponding to $B(G, \mu) \subset B(G)$, following the main result in [RR96]. Since the natural projection $B(P) \to B(G)$ is a bijection ([Kot97, §3.6]), we let $P\text{-}\mathrm{Isoc}_\mu \subset P\text{-}\mathrm{Isoc}$ be the pullback of $G\text{-}\mathrm{Isoc}_\mu \subset G\text{-}\mathrm{Isoc}$, which is again a closed substack.

Let $\mathrm{Sht}_{\mathcal{P}}^W$ be the functor sending any perfect algebra R to the groupoid of $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $\mathcal{P}$-torsor on $\operatorname{Spec} W(R)$ and $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$ a modification. By [PR24, Thm. 2.3.8] and the Tannakian formalism, there is a natural isomorphism $\mathrm{Sht}_{\mathcal{P}}^{\mathrm{red}} \cong \mathrm{Sht}_{\mathcal{P}}^W$; cf. [DvHKZ24, Lem. 3.1.5]. When $\mathcal{P}$ is parahoric, one can define $\mathrm{Sht}_{\mathcal{P},\mu}^W$ by requiring the relative position to be controlled by $\mathrm{Gr}_{\mathcal{P},\mu}^W$. Then, under $\mathrm{Sht}_{\mathcal{P}}^{\mathrm{red}} \cong \mathrm{Sht}_{\mathcal{P}}^W$, we have $\mathrm{Sht}_{\mathcal{P},\mu}^{\mathrm{red}} \cong \mathrm{Sht}_{\mathcal{P},\mu}^W$ by [DvHKZ24, Lem. 3.1.7] (with the help of Corollary 1.30). When $\mathcal{P}$ is quasi-parahoric, we let $\mathrm{Sht}_{\mathcal{P},\mu}^W \subset \mathrm{Sht}_{\mathcal{P}}^W$ be the reduction of $\mathrm{Sht}_{\mathcal{P},\mu} \subset \mathrm{Sht}_{\mathcal{P}}$; this is compatible with the parahoric case.

Recall that the natural projection $\mathrm{Sht}_{\mathcal{P}}^W \to P\text{-}\mathrm{Isoc}$ defined by $(\mathcal{E}, \beta) \mapsto (\mathcal{E}|_{W(*)[1/p]}, \beta)$ is a v-cover. By construction, $\mathrm{Sht}_{\mathcal{P}}^W \to P\text{-}\mathrm{Isoc} \to G\text{-}\mathrm{Isoc}$ factors through $\mathrm{Sht}_{\mathcal{G}}^W$. By [DvHKZ24, §3.2.2], the projection $\mathrm{Sht}_{\mathcal{G},\mu}^W \to G\text{-}\mathrm{Isoc}$ factors through $G\text{-}\mathrm{Isoc}_{\mu^{-1}}$ when $\mathcal{G}$ is parahoric, then $\mathrm{Sht}_{\mathcal{P},\mu}^W \to P\text{-}\mathrm{Isoc}$ factors through $P\text{-}\mathrm{Isoc}_{\mu^{-1}}$ when $\mathcal{P}$ is parahoric.

Recall that one has the Beauville–Laszlo morphism $\mathrm{BL} : \mathrm{Sht}_{\mathcal{P}} \to \mathrm{Bun}_P$ as follows: let $(\mathscr{P}, \phi_{\mathscr{P}})$ be a $\mathcal{P}$-shtuka on S with one leg at $S^\sharp$. Restrict $(\mathscr{P}, \phi_{\mathscr{P}})$ to $\mathcal{Y}_{[r,\infty)}(S)$ for sufficiently large r so that it excludes the leg at $S^\sharp$, and descend it to $X_{\mathrm{FF},S}$; we obtain a $\mathcal{P}$-torsor $\mathcal{E}(\mathscr{P}, \phi_{\mathscr{P}})$ on $X_{\mathrm{FF},S}$. By taking the reduction, we have a morphism $\mathrm{BL}^{\mathrm{red}} : \mathrm{Sht}_{\mathcal{P}}^{\mathrm{red}} \to \mathrm{Bun}_P^{\mathrm{red}}$. This is compatible with the projection $\mathrm{Sht}_{\mathcal{P}}^W \to P\text{-}\mathrm{Isoc}$ together with the identification $P\text{-}\mathrm{Isoc} \cong \mathrm{Bun}_P^{\mathrm{red}}$.

On the other hand, we have a functor $P\text{-}\mathrm{Isoc} \to \mathrm{Bun}_P$, defined by composing the exact $\otimes$-functor $\mathrm{Rep}_{\mathbb{Q}_p} P \to \mathrm{Isoc}$ with the exact $\otimes$-functor $\mathrm{Isoc} \to \mathrm{Bun}$ (see [FS21, §III.2], [Ans19]). By results in [Far20] and [Ans19], when $P = G$ is a reductive group, such a functor induces a bijection on the underlying topological spaces (and moreover a homeomorphism when $B(G)$ is endowed with the topology induced by the Bruhat order reversed from [RR96]; see [Vie24]). In general, we also have

Lemma 1.44. *The functor* $P\text{-}\mathrm{Isoc} \to \mathrm{Bun}_P$ *induces a bijection* $B(P) \to \mathrm{Bun}_P(C)/\cong$.

Proof. Since $B(P) \to B(G)$ and $B(G) \to \mathrm{Bun}_G(C)/\cong$ are bijections, it suffices to show that $H^1(X_{\mathrm{FF},C}, U) = 0$. In characteristic 0, the unipotent group U is split and is a successive extension of $\mathbb{G}_{\mathrm{a}}$. Since $H^1(X_{\mathrm{FF},C}, \mathbb{G}_{\mathrm{a}}) = H^1(X_{\mathrm{FF},C}, \mathcal{O}_{X_{\mathrm{FF},C}}) = 0$ (see [FS21, Prop. II.2.5(ii)]), we have $H^1(X_{\mathrm{FF},C}, U) = 0$. □

1.3.4. *Shtukas and local systems.* Using the Tannakian formalism and Beauville–Laszlo gluing, and following the proofs of [SW20, Prop. 12.4.6] and [PR24, Prop. 2.5.1] ([SW20, Thm. 22.5.2, Prop. 22.6.1] works for any smooth affine model $\mathcal{P}$ with connected fibers), we have the following:

Proposition 1.45 ([PR24, Prop. 2.5.1])**.** *Let* $\mathcal{P}$ *be parahoric. The construction*

$$(\mathscr{P}, \phi_{\mathscr{P}}) \mapsto (\mathbb{P}, \mathrm{DRT}(\mathscr{P})) \tag{1.10}$$

gives an equivalence of categories between

1. *$\mathcal{P}$-shtukas over $S/\operatorname{Spd} E$ (resp. with one leg bounded by μ),*
2. *pairs $(\mathbb{P}, D)$, where $\mathbb{P}$ is a pro-étale $\underline{\mathcal{P}(\mathbb{Z}_p)}$-torsor over S and $D : \mathbb{P} \to \mathrm{Gr}_{P,\operatorname{Spd} E}$ (resp. $D : \mathbb{P} \to \mathrm{Gr}_{P,\mu^{-1}}$) is a $\underline{\mathcal{P}(\mathbb{Z}_p)}$-equivariant morphism over $\operatorname{Spd} E$.*

This result can be generalized to the case where the base $S/\operatorname{Spd} E$ is replaced by a v-sheaf $\mathcal{F}$.

Finally, as in [PR24, §2.6], using the Tannakian formalism, given a de-Rham pro-étale torsor $\mathbb{P}$ under $\underline{\mathcal{P}(\mathbb{Z}_p)}$ over $X^\Diamond / \operatorname{Spd} E$ (resp. we moreover assume that for all classical points $x \in X$, the filtration $\operatorname{Fil}^\bullet D_{\mathrm{dR}}(\mathbb{L}_{\rho,\bar{x}})$ has constant conjugacy class), then $\mathbb{P}$ is equipped with a canonical Hodge–Tate period map $D : \mathbb{P} \to \mathrm{Gr}_{P,\operatorname{Spd} E}$ (resp. $D : \mathbb{P} \to \mathrm{Gr}_{P,\mu^{-1}}$), and one can attach to it a $\mathcal{P}$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ over $X^\Diamond / \operatorname{Spd} E$ (resp. with one leg bounded by μ).

1.3.5. *Quasi-parahoric group schemes.*

Lemma 1.46. *Let $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$ be a quasi-parahoric group scheme. Then the following 2-commutative diagrams are 2-Cartesian:*

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{P}^\circ} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}} \arrow[d] & \mathrm{Sht}_{\mathcal{P}^\circ,\mu} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P},\mu} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}^\circ} \arrow[r] & \mathrm{Sht}_{\mathcal{G}}, & \mathrm{Sht}_{\mathcal{G}^\circ,\mu} \arrow[r] & \mathrm{Sht}_{\mathcal{G},\mu}.
\end{tikzcd}$$

Proof. Over $S \dot{\times} \mathbb{Z}_p$, it is easy to show that the groupoid of $\mathcal{P}^\circ$-torsors is isomorphic to the groupoid of $\mathcal{P}$-torsors whose reduction to $\mathcal{G}$-torsors comes from $\mathcal{G}^\circ$-torsors. To check the boundedness condition, we use Lemma 1.25. ☐

Consider the Kottwitz map $\kappa_G : |\operatorname{Bun}_G| \to \pi_1(G)_\Gamma$, where $\Gamma = \operatorname{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ and $\pi_1(G)$ is the algebraic fundamental group of G. Then κ_G is locally constant and maps $\operatorname{Bun}_{G,\mu^{-1}}$ to $-\mu^\natural$. Let $\mathcal{G}$ be a quasi-parahoric group scheme of G, and let $\mathrm{Sht}_{\mathcal{G},\mu}^{\kappa=-\mu^\natural} \subset \mathrm{Sht}_{\mathcal{G},\mu}$ be the open and closed substack that maps to $-\mu^\natural$ under the composition of the Beauville–Laszlo map $\mathrm{BL}^\circ : \mathrm{Sht}_{\mathcal{G},\mu} \to \operatorname{Bun}_G$ with κ_G. The map BL° does not factor through $\operatorname{Bun}_{G,\mu^{-1}}$; nevertheless, the restriction $\mathrm{BL}^\circ : \mathrm{Sht}_{\mathcal{G},\mu}^{\kappa=-\mu^\natural} \to \operatorname{Bun}_G$ does factor through $\operatorname{Bun}_{G,\mu^{-1}}$; see [DvHKZ26, Prop. 3.1.10].

Recall that we have a short exact sequence

$$1 \to \mathcal{G}^\circ(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}_p) \to \pi_0(\mathcal{G})^\phi \to 1.$$

In [DvHKZ26, §3.2], there is an action of $\pi_0(\mathcal{G})^\phi$ on $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$, and the pushout $\mathcal{E} \mapsto \mathcal{G} \times^{\mathcal{G}^\circ} \mathcal{E}$ naturally induces an open and closed substack $[\mathrm{Sht}_{\mathcal{G}^\circ,\mu}/\pi_0(\mathcal{G})^\phi] \to \mathrm{Sht}_{\mathcal{G},\mu}$; see [DvHKZ26, Thm. 3.3.5]. Let $\mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ be the image of $\mathrm{Sht}_{\mathcal{G}^\circ,\mu}$. Then $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}$ is a torsor under the finite abelian group $\pi_0(\mathcal{G})^\phi$.

Let $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$ be a quasi-parahoric group scheme. We define

$$\mathrm{Sht}_{\mathcal{P},\mu,\delta=1} \subset \mathrm{Sht}_{\mathcal{P},\mu}^{\kappa=-\mu^\natural} \subset \mathrm{Sht}_{\mathcal{P},\mu} \quad (\textit{resp.}\ \mathrm{Sht}_{\mathcal{P},\mu,\delta=1}^{W} \subset \mathrm{Sht}_{\mathcal{P},\mu}^{W,\kappa=-\mu^\natural} \subset \mathrm{Sht}_{\mathcal{P},\mu}^{W})$$

as the open and closed substacks that are the pullbacks of

$$\mathrm{Sht}_{\mathcal{G},\mu,\delta=1} \subset \mathrm{Sht}_{\mathcal{G},\mu}^{\kappa=-\mu^\natural} \subset \mathrm{Sht}_{\mathcal{G},\mu} \quad (\textit{resp.}\ \mathrm{Sht}_{\mathcal{G},\mu,\delta=1}^{W} \subset \mathrm{Sht}_{\mathcal{G},\mu}^{W,\kappa=-\mu^\natural} \subset \mathrm{Sht}_{\mathcal{G},\mu}^{W})$$

under the natural projection $\mathrm{Sht}_{\mathcal{P},\mu} \to \mathrm{Sht}_{\mathcal{G},\mu}$ (resp. $\mathrm{Sht}_{\mathcal{P},\mu}^{W} \to \mathrm{Sht}_{\mathcal{G},\mu}^{W}$). Then $\mathrm{Sht}_{\mathcal{P},\mu}^{W,\kappa=-\mu^\natural} \to P\text{-}\mathrm{Isoc}$ factors through $P\text{-}\mathrm{Isoc}_{\mu^{-1}}$.

Also, we have a short exact sequence

$$1 \to \mathcal{P}^\circ(\mathbb{Z}_p) \to \mathcal{P}(\mathbb{Z}_p) \to \pi_0(\mathcal{P})^\phi \to 1. \tag{1.11}$$

We define an action of $\pi_0(\mathcal{P})^\phi$ on $\mathrm{Sht}_{\mathcal{P}^\circ,\mu}$ as in [DvHKZ26, §3.2], and the pushout functor naturally induces $[\mathrm{Sht}_{\mathcal{P}^\circ,\mu}/\pi_0(\mathcal{P})^\phi] \to \mathrm{Sht}_{\mathcal{P},\mu}$.

Corollary 1.47 (cf. [DvHKZ26, Cor. 3.3.8, 3.3.9, Rmk. 4.1.5])**.** *(1) $[\mathrm{Sht}_{\mathcal{P}^\circ,\mu}/\pi_0(\mathcal{P})^\phi]$ is open and closed in $\mathrm{Sht}_{\mathcal{P},\mu}$ with image $\mathrm{Sht}_{\mathcal{P},\mu,\delta=1}$. In particular, $\mathrm{Sht}_{\mathcal{P}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{P},\mu,\delta=1}$ is a torsor under the finite abelian group $\pi_0(\mathcal{P})^\phi$.*

(2) There is a natural isomorphism

$$\mathrm{Sht}_{\mathcal{P},\mu,\delta=1}\times_{\mathrm{Spd}\,\mathcal{O}_E}\mathrm{Spd}\,E\cong[\mathrm{Gr}_{P,\mu^{-1}}/\underline{\mathcal{P}(\mathbb{Z}_p)}].$$

(3) Let X be a normal scheme of finite type and flat over $\mathbb{Z}_p$. Assume that we have $X^{\Diamond/}\to\mathrm{Sht}_{\mathcal{P},\mu}$, and $X_\eta^\Diamond\to\mathrm{Sht}_{\mathcal{P},\mu,\mathrm{Spd}\,E}$ factors through $\mathrm{Sht}_{\mathcal{P},\mu,\delta=1,\mathrm{Spd}\,E}$. Then $X^{\Diamond/}\to\mathrm{Sht}_{\mathcal{P},\mu}$ factors through $\mathrm{Sht}_{\mathcal{P},\mu,\delta=1}$.

Proof. (1) By definition, $\pi_0(\mathcal{G})=\pi_0(\mathcal{P})$. This corollary directly follows from Lemma 1.46.

(2) For parahoric subgroups, this follows from [Zha26, Prop. 11.16]; note that [SW20, Thm. 22.5.2, Prop. 22.6.1] works for any smooth affine model $\mathcal{P}$ with connected fibers. For quasi-parahoric subgroups, apply the results in Part (1).

(3) See [DvHKZ26, Rmk. 4.1.5].

□

Lemma 1.48. *Let $P_1=U\rtimes G_1$, $P_2=U\rtimes G_2$, $P_1\to P_2$ be a morphism compatible with $U\rtimes G_1\xrightarrow{(\mathrm{id},f)}U\rtimes G_2$ in the sense of 1.33. Let $\mathcal{P}_i=\mathcal{U}\rtimes\mathcal{G}_i$ be parahoric group schemes of P_i. Assume $P_1\to P_2$ induces $\mathcal{P}_1\to\mathcal{P}_2$ that is compatible with $\mathcal{G}_1\to\mathcal{G}_2$ in the sense of 1.35. Assume the cocharacter μ of P_1 is quotient-minuscule in both P_1 and P_2. Then the following 2-commutative diagrams are 2-Cartesian:*

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{P}_1} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_2} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}_1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sht}_{\mathcal{P}_1,\mu} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_2,\mu} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}_1,\mu} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2,\mu}.
\end{tikzcd}$$

Proof. The left diagram is Cartesian; see [DY25, Cor. 2.17]. Note that $\mathcal{P}_1=\mathcal{G}_1\times_{\mathcal{G}_2}\mathcal{P}_2$ by assumption. The right diagram is Cartesian by Lemma 1.38. □

Corollary 1.49. *Keep the notation and assumptions from Lemma 1.48, with the word **parahoric** replaced by **quasi-parahoric**. Then the following 2-commutative diagram is 2-Cartesian:*

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{P}_1,\mu,\delta=1} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_2,\mu,\delta=1} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}_1,\mu,\delta=1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2,\mu,\delta=1}.
\end{tikzcd} \tag{1.12}$$

Proof. First, assume $G_1=G_2$ and $\mathcal{G}_1\to\mathcal{G}_2$ induces $\mathcal{G}_1^\circ=\mathcal{G}_2^\circ$. By [DvHKZ26, Cor. 3.3.11] and the first part of Corollary 1.47, the diagram (1.12) is 2-Cartesian, where both horizontal morphisms are torsors under finite abelian groups

$$\pi_0(\mathcal{P}_2)^\phi/\pi_0(\mathcal{P}_1)^\phi=\pi_0(\mathcal{G}_2)^\phi/\pi_0(\mathcal{G}_1)^\phi.$$

In general, given a morphism of quasi-parahoric group schemes $\mathcal{G}_1\to\mathcal{G}_2$, then $\mathcal{G}_1^\circ\to\mathcal{G}_1\to\mathcal{G}_2$ factors through $\mathcal{G}_2^\circ$, this can be seen by the equation $\mathcal{G}_i^\circ(\breve{\mathbb{Z}}_p)=\mathcal{G}_i(\breve{\mathbb{Z}}_p)\cap\ker\tilde{\kappa}_{G_i}$, where $\tilde{\kappa}_{G_i}:G_i(\breve{\mathbb{Q}}_p)\to\pi_1(G_i)_I$ is the functorial Kottwitz map. Consider the diagram

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{P}_1^\circ,\mu} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Sht}_{\mathcal{P}_2^\circ,\mu} \arrow[dr] \arrow[dd] & \\
& \mathrm{Sht}_{\mathcal{P}_1,\mu,\delta=1} \arrow[rr] \arrow[dd] & & \mathrm{Sht}_{\mathcal{P}_2,\mu,\delta=1} \arrow[dd] \\
\mathrm{Sht}_{\mathcal{G}_1^\circ,\mu} \arrow[rr] \arrow[dr] & & \mathrm{Sht}_{\mathcal{G}_2^\circ,\mu} \arrow[dr] & \\
& \mathrm{Sht}_{\mathcal{G}_1,\mu,\delta=1} \arrow[rr] & & \mathrm{Sht}_{\mathcal{G}_2,\mu,\delta=1}.
\end{tikzcd}$$

Since the left, right and the back diagrams are 2-Cartesian by above discussions and Lemma 1.48, and since $\mathrm{Sht}_{\mathcal{G}_1^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G}_1,\mu,\delta=1}$ is a surjection of v-stacks, the front diagram (1.12) is 2-Cartesian. □

2. LOG DIAMONDS AND LOG SHTUKAS

We discuss some theorems in [PR24, Sec. 2] in the log setting. To start with, we also introduce some basic results about log diamonds defined in the v-topology for log adic spaces, schemes and formal schemes. Since we will deal with many slightly different cases simultaneously, the readers will be frequently reminded about the cases under consideration before seeing the results.

2.1. Log diamonds associated with log schemes.

Our goal is to develop results on log diamonds associated with fs log schemes over $\mathbb{Z}_p$. This may be viewed as a common generalization of some definitions and propositions in [SW20, §18] and [KY25, §7].

2.1.1. Let X be a scheme locally of finite type over $\mathbb{Z}_p$. We write X^{ad} as the adic space that represents the fiber product

$$X \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spa}\,\mathbb{Z}_p.$$

Let $\widehat{X}$ be the formal scheme defined by the p-adic completion of X and $\widehat{X}^{\mathrm{ad}}$ the adic space associated with it. We can also define X^{ad} similarly when X is of finite type over $\mathbb{Q}_p$.

When X is separated and of finite type, there is an open embedding $\widehat{X}^{\mathrm{ad}} \hookrightarrow X^{\mathrm{ad}}$. Now let X be a separated scheme of finite type over $\mathcal{O}_E$, where E is a finite field extension of $\mathbb{Q}_p$. We list some easy facts that will be used later:

(1) Let X_η be the generic fiber of X, we can also define $(X_\eta)^{\mathrm{ad}} = X_\eta \times_{\mathrm{Spec}\,E} \mathrm{Spa}(E, \mathcal{O}_E)$ in the sense of [Hub94, Proposition 3.8], then

$$(X_\eta)^{\mathrm{ad}} = (X \times_{\mathrm{Spec}\,\mathcal{O}_E} \mathrm{Spa}\,\mathcal{O}_E) \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}(E, \mathcal{O}_E) = (X^{\mathrm{ad}})_\eta. \tag{2.1}$$

(2) Let $X \to Y \leftarrow Z$ be schemes separated locally of finite type over E, then X^{ad}, Y^{ad}, Z^{ad} are Tate. In particular, $X^{\mathrm{ad}} \to Y^{\mathrm{ad}} \leftarrow Z^{\mathrm{ad}}$ are adic, the fiber product exists. Checking the functoriality in [Hub94, Proposition 3.8], we have

$$(X \times_Y Z)^{\mathrm{ad}} = X^{\mathrm{ad}} \times_{Y^{\mathrm{ad}}} Z^{\mathrm{ad}} \tag{2.2}$$

(3) Let X be a smooth variety over E, then X^{ad} is a sousperfectoid analytic adic space.

(4) Let $\widehat{X} \to \widehat{Y} \leftarrow \widehat{Z}$ be formal schemes separated locally of finite type over $\mathcal{O}_E$, then (2.2) also holds:

$$(\widehat{X} \times_{\widehat{Y}} \widehat{Z})^{\mathrm{ad}} = \widehat{X}^{\mathrm{ad}} \times_{\widehat{Y}^{\mathrm{ad}}} \widehat{Z}^{\mathrm{ad}}. \tag{2.3}$$

(5) The functor $X^{\Diamond}$ (resp. $X^{\diamond}$) can be constructed via taking the diamond functor for adic spaces for X^{ad} (resp. $\widehat{X}^{\mathrm{ad}}$); both $(-)^{\Diamond}$ and $(-)^{\diamond}$ commute with fiber products in the category of separated schemes of finite type over $\mathcal{O}_E$, and are compatible with (2.2) and (2.3) under (2.1).

2.1.2. Let us recall some terminology in log geometry. For details, we refer the readers to [Kat89], [Ogu18], [DLLZ23a] and [KY25].

Let P be a monoid. One can associate a group P^{gp} with a natural homomorphism between monoids $\mathrm{gp}_{\mathsf{P}} : \mathsf{P} \to \mathsf{P}^{\mathrm{gp}}$; in fact, $\mathsf{P} \mapsto \mathsf{P}^{\mathrm{gp}}$ is the left adjoint functor of the natural inclusion of the category of groups into the category of monoids.

We say that P is *integral* if gp_{P} is injective, say that P is *saturated*, if P is integral and any $x \in \mathsf{P}^{\mathrm{gp}}$ such that $nx \in \mathsf{P}$ for some $n \in \mathbb{Z}_{>0}$ is in P, and say that P is *fine* if P is integral and finitely generated. We say that P is *fs* if P is both fine and saturated. Denote $\overline{\mathsf{P}} := \mathsf{P}/\mathsf{P}^{\times}$.

We say an adic space X is *étale sheafy* if $X_{\text{ét}}$ is a site and if $\mathcal{O}_{X_{\text{ét}}}$ is a sheaf. This is satisfied, for example, when X is analytic stably sheafy or is Noetherian affinoid (cf. [DLLZ23a, Cor. A.11]).

Definition 2.1 ([DLLZ23a, Def. 2.2.2])**.** *Let X be an étale sheafy adic space. A* prelog structure *on X is a pair $(\mathcal{M}_X, \alpha)$, where $\mathcal{M}_X$ is a sheaf of monoids over $X_{\text{ét}}$ and $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}}$ is a morphism of sheaves of monoids, and such a pair is a* log structure *if the induced morphism $\alpha^{-1}(\mathcal{O}^\times_{X_{\text{ét}}}) \to \mathcal{O}^\times_{X_{\text{ét}}}$ is an isomorphism.*

A log adic space *is a triple $(X, \mathcal{M}_X, \alpha)$ consisting of an étale sheafy adic space X and a log structure $(\mathcal{M}_X, \alpha)$ as above.*

A morphism *$f : (X, \mathcal{M}_X, \alpha_X) \to (Y, \mathcal{M}_Y, \alpha_Y)$ of log adic spaces is a morphism $f : X \to Y$ of underlying adic spaces X and Y with a morphism of sheaves of monoids $f^\sharp : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the diagram*

$$\begin{tikzcd}
f^{-1}\mathcal{M}_Y \arrow[r, "f^\sharp"] \arrow[d] & \mathcal{M}_X \arrow[d] \\
f^{-1}\mathcal{O}_{Y_{\text{ét}}} \arrow[r] & \mathcal{O}_{X_{\text{ét}}}
\end{tikzcd} \tag{2.4}$$

commutes.

We call such a log structure $(\mathcal{M}_X, \alpha)$ a log structure on $X_{\text{ét}}$ (or simply on X). We omit α if it is not important or is clear in the context.

Definition 2.2.
1. *Let $\mathcal{M}$ be a log structure on $X_{\text{ét}}$. We say $\mathcal{M}_X$ is* integral *(resp.* saturated*) if it is a sheaf of integral (resp. saturated) monoids. Let $f : X \to Y$ be a morphism of adic spaces. The* inverse image *$f^*\mathcal{M}_Y$ of a log structure $\mathcal{M}_Y$ of Y is defined to be the log structure associated with the prelog structure on X given by $f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_{Y_{\text{ét}}} \to \mathcal{O}_{X_{\text{ét}}}$ (cf.* [Kat89, (1.4)]*).*
2. *For a sheaf of monoids $\mathcal{M}$ on $X_{\text{ét}}$, denote by $\mathcal{M}^\times$ its sheaf of invertible elements. Denote by $\mathcal{M}^\flat$ its tilt $\varprojlim_{x\mapsto x^p} \mathcal{M}$. Following* [KY25, Def. 2.19]*, we say a sheaf of monoids $\mathcal{M}$ on $X_{\text{ét}}$ is* perfectoid *if $\overline{\mathcal{M}^\flat} := \mathcal{M}^\flat/\mathcal{M}^{\flat,\times} \to \mathcal{M}/\mathcal{M}^\times =: \overline{\mathcal{M}}$ is an isomorphism. In particular, $\overline{\mathcal{M}}$ is* uniquely p-divisible *if $\mathcal{M}$ is perfectoid (cf.* [KY25, Rmk. 2.23]*).*

Definition 2.3 ([DLLZ23a, Def. 2.3.1])**.** *Let P be a monoid. A* chart of $(X, \mathcal{M}_X)$ modeled on P *is a morphism $\theta : \mathsf{P}_X \to \mathcal{M}_X$ of étale sheaves of monoids such that $\alpha \circ \theta$ factors through $\mathcal{O}^+_{X_{\text{ét}}}$ and the associated log structure of $\alpha \circ \theta$ is canonically isomorphic to that of α. We say the chart is fs if P is fs.*

A log adic space X is *fs* (resp. *fine*) if X étale locally admits fs (resp. fine) charts. Following the conventions in [Kat89], for a prelog ring/space (A, P), one can associate a log ring/space denoted by $(A, \mathsf{P})^a$ (see also [DLLZ23a, Def. 2.2.2(6)]); the functor $(-)^a$ is the left adjoint of the natural inclusion of the category of log adic spaces into the category of prelog adic spaces.

2.1.3. *The cases we consider.* From now on, the pair $(X, \mathcal{M}_X)$ can be the following cases:
1. an fs log adic space over $\operatorname{Spa} \mathbb{Z}_p$;
2. an fs log scheme over $\operatorname{Spec} \mathbb{Z}_p$;
3. an fs log formal scheme over $\operatorname{Spf} \mathbb{Z}_p$.

As a (pre-)adic space might not be étale sheafy in general, sometimes it is worth separating the last two cases.

Let X be a scheme over $\mathbb{Z}_p$ or $\mathbb{Q}_p$. In some situations, we assume that

(SF): Let $(X, \mathcal{M}_X)$ be an fs log scheme over $\mathbb{Z}_p$ or $\mathbb{Q}_p$ with X a separated and of finite type over $\mathbb{Z}_p$ or $\mathbb{Q}_p$.

We do not assume the properness of $X \to \operatorname{Spec} \mathbb{Z}_p$ here.

By [DLLZ23a, Cor. A.11], for log schemes satisfying (SF), X^{ad} and $\widehat{X}^{\text{ad}}$ are both étale sheafy.

There is a natural morphism of étale sites $\nu^{\mathrm{ad}} := \nu_X^{\mathrm{ad}} : X_{\text{ét}}^{\mathrm{ad}} \to X_{\text{ét}}$ (see [Hub96, 3.2.8, p.180]). Also, there is a natural morphism of étale sites $\widehat{\nu} := \widehat{\nu}_X : \widehat{X}_{\text{ét}}^{\mathrm{ad}} \to X_{\text{ét}}$ defined by composing with $\widehat{X}_{\text{ét}}^{\mathrm{ad}} \to X_{\text{ét}}^{\mathrm{ad}}$.

2.1.4. To define log diamonds, it is crucial to study a certain class of log structures on perfectoid spaces.

Let Y be a perfectoid space. Denote by $Y_{\text{ét}}$ (resp. Y_v) the étale site (resp. v-site) of Y. There is a natural projection of sites

$$\nu : Y_v \to Y_{\text{ét}}.$$

By [Sch26, Thm. 8.7], $\mathcal{O}_{Y_v}$ is a v-sheaf. So it makes sense to define log structures on Y_v.

Definition 2.4. *A prelog structure on Y_v is a pair $(\mathcal{M}_Y, \alpha)$ consisting of a v-sheaf of monoids $\mathcal{M}_Y$ in $Y_v^\sim$ and a morphism between v-sheaves $\alpha : \mathcal{M}_Y \to \mathcal{O}_{Y_v}$. It is a log structure on Y_v if $\alpha^{-1}(\mathcal{O}_{Y_v}^\times) \to \mathcal{O}_{Y_v}^\times$ is an isomorphism.*

The definitions of morphisms and charts are also similar to those defined for log structures on $Y_{\text{ét}}$ (cf. Definition 2.1 and 2.3).

From now on, we let Y_τ stand for $Y_{\text{ét}}$ (resp. Y_v) if $\tau = \text{ét}$ (resp. $\tau = v$). We say a sheaf of monoids $\mathcal{M}$ on Y_τ is uniquely p-divisible (resp. perfectoid) if $\mathcal{M} \xrightarrow{x \mapsto x^p} \mathcal{M}$ (resp. $\overline{\mathcal{M}^\flat} \to \overline{\mathcal{M}}$) is an isomorphism. That is, for any $U \in Y_\tau$, $\mathcal{M}(U)$ is uniquely p-divisible (resp. $\overline{\mathcal{M}^\flat}(U) \to \overline{\mathcal{M}}(U)$ is an isomorphism).

The following property is weaker than being uniquely n-divisible but stronger than being n-torsion-free.

Definition 2.5. *Let n be a positive integer and P a monoid. We say that P is n-**unique** if $\mathsf{P} \xrightarrow{n\cdot} \mathsf{P}$ is injective. Similarly, a sheaf of monoids $\mathcal{M}$ on a site $\mathcal{C}$ is n-unique if $\mathcal{M}(U)$ is n-unique for any $U \in \mathrm{Ob}\,\mathcal{C}$.*

Definition 2.6. *Let Y be a perfectoid space. Set $\tau = \text{ét}$ or v. Let $\mathbb{N}[\frac{1}{p}] \subset \mathbb{Q}$ be the monoid consisting of elements of the form $\frac{a}{b}$ where $a \in \mathbb{N}$ and $b \in \{p^n\}$ for $n \geq 0$.*

1. *A uniquely p-divisible monoid P is called p-**finitely generated** if P is n-unique for all positive integers n, and there is a finite set of elements $S \subset \mathsf{P}$ such that the set of all p^i-th roots of elements of S generates P for integers $i > 0$. In other words, there is a surjection of monoids $\mathbb{N}[\frac{1}{p}]^{\oplus n} \twoheadrightarrow \mathsf{P}$.*
2. *A uniquely p-divisible monoid P is called p-**weakly finitely generated** if P is n-unique for all positive integers n, and there is a uniquely p-divisible, p-finitely generated submonoid $\mathsf{P}' \subset \mathsf{P}$ such that $\mathsf{P}' \subset \mathsf{P} \subset \mathbb{Q}_{\geq 0}\mathsf{P}'$. Note that the notation $\mathbb{Q}_{\geq 0}\mathsf{P}' := \varinjlim_{a \mapsto na, a \in \mathsf{P}, n \geq 1} \mathsf{P}$ and the inclusions make sense because of the n-uniqueness assumption.*
3. *A perfectoid log structure $(\mathcal{M}_Y, \alpha)$ on Y_τ is p-**coherent** (resp. p-**weakly coherent**) if, for any geometric point $\overline{x}$ of $Y_{\text{ét}}$, there is an étale neighborhood $U_{\overline{x}}$ such that $\mathcal{M}_Y|_{U_{\overline{x}}}$ admits a p-finitely generated (resp. p-weakly finitely generated) chart $\mathsf{P}_{U_{\overline{x}}}$ (of log structures on $(U_{\overline{x}})_\tau$).*
4. *A log structure $(\mathcal{M}_Y, \alpha)$ on Y_τ is called **fine perfectoid** (resp. strongly fine perfectoid) if $\mathcal{M}_Y$ is integral, perfectoid, and p-weakly coherent (resp. p-coherent).*

Remark 2.7. *A log structure $\mathcal{M}$ on Y_v is fine perfectoid implies that $\nu_*\mathcal{M}$ on $Y_{\text{ét}}$ is fine perfectoid. In fact, to see that $\nu_*\mathcal{M}$ is perfectoid, note that $\overline{\nu_*\mathcal{M}} \cong \nu_*\overline{\mathcal{M}}$ and $\overline{\nu_*\mathcal{M}^\flat} \cong \nu_*\overline{\mathcal{M}^\flat}$; this follows from Lemma 2.25. Then the isomorphism of sheaves holds after pushforward.*

Remark 2.8. *We remark that the integrality is crucial in the proof of Theorem 2.18, while perfectoidness and (weak) p-finite generation will be used in the proof of Theorem 2.49.*

We discuss some basic properties of the log structures defined above.

Lemma 2.9 (cf. [DLLZ23a, Lem. 2.1.10])**.** *Let $\lambda : \mathsf{M} \to \mathsf{P}'$ be a surjective morphism between integral monoids such that $\ker \lambda^{\mathrm{gp}} \subset \mathsf{M}$. Suppose that P' is additionally saturated, sharp, uniquely p-divisible, and p-finitely generated monoid. Then λ admits a section.*

Proof. By integrality, there are injections $\mathsf{M} \hookrightarrow \mathsf{M}^{\mathrm{gp}}$ and $\mathsf{P}' \hookrightarrow \mathsf{P}'^{,\mathrm{gp}}$. Since P' is saturated and sharp, $\mathsf{P}'^{,\mathrm{gp}}$ is torsion-free. As there is a surjection $\mathbb{N}[\frac{1}{p}]^N \to \mathsf{P}'$ for some positive integer N, we know that there is a surjection $\mathbb{Z}[\frac{1}{p}]^N \to \mathsf{P}'^{,\mathrm{gp}}$. So $\mathsf{P}'^{,\mathrm{gp}} \cong \mathbb{Z}[\frac{1}{p}]^n$ for some $n \leq N$ as $\mathbb{Z}[1/p]$ is a PID. So $\lambda^{\mathrm{gp}} : \mathsf{M}^{\mathrm{gp}} \to \mathsf{P}'^{,\mathrm{gp}}$ admits a section. Since $\ker \lambda^{\mathrm{gp}} \subset \mathsf{M}$, we see this section restricted to P' must be contained in M. In fact, for any $s \in \mathsf{M}$, $\lambda^{\mathrm{gp},-1}(\lambda(s)) = \lambda^{-1}(\lambda(s)) \subset \mathsf{M}$. ☐

Lemma 2.10. *Let $(\mathcal{M}_Y, \alpha)$ be a saturated and fine perfectoid log structure on Y_τ. Then $\overline{\mathcal{M}}_Y$ is saturated, uniquely p-divisible, and p-weakly finitely generated.*

Proof. We only show it when $\tau =$ ét. The case when $\tau = v$ is similar. Saturatedness and unique p-divisibility follow from the saturatedness and the fine-perfectoidness of $\mathcal{M}_Y$, respectively. The question is étale local, so we assume that $\mathcal{M}_Y$ admits a global (uniquely p-divisible and) p-weakly finitely generated chart P. From the definition above, there is a p-finitely generated submonoid P' and inclusions $\mathsf{P}' \subset \mathsf{P} \subset \mathbb{Q}_{\geq 0}\mathsf{P}'$. Fix a surjection $\mathbb{N}[\frac{1}{p}]^n \twoheadrightarrow \mathsf{P}'$ with a commutative diagram

$$\begin{tikzcd}
\mathbb{N}[\frac{1}{p}]^n \arrow[r, hook] \arrow[d, two heads] & \mathsf{N} \arrow[r, hook] \arrow[d, two heads] & \mathbb{Q}^n_{\geq 0} \arrow[d, two heads] \\
\mathsf{P}' \arrow[r, hook] & \mathsf{P} \arrow[r, hook] & \mathbb{Q}_{\geq 0}\mathsf{P}',
\end{tikzcd}$$

where N is the pullback of P via $\mathbb{Q}^n_{\geq 0} \to \mathbb{Q}_{\geq 0}\mathsf{P}'$.

Let $\overline{\mathcal{M}}'_Y$ be the image of P'_Y via $\mathsf{P}_Y \to \overline{\mathcal{M}}_Y$. The diagram above extends to a commutative diagram

$$\begin{tikzcd}
\mathbb{N}[\frac{1}{p}]^n_Y \arrow[r, hook] \arrow[d, two heads] & \mathsf{N}_Y \arrow[r, hook] \arrow[d, two heads] & \mathbb{Q}^n_{\geq 0, Y} \arrow[d, two heads] \\
\mathsf{P}'_Y \arrow[r, hook] \arrow[d, two heads] & \mathsf{P}_Y \arrow[r, hook] \arrow[d, two heads] & (\mathbb{Q}_{\geq 0}\mathsf{P}')_Y \\
\overline{\mathcal{M}}'_Y \arrow[r, hook] & \overline{\mathcal{M}}_Y. &
\end{tikzcd} \tag{2.5}$$

Fix $U \in Y_{\text{ét}}$ and $x_1, x_2 \in \overline{\mathcal{M}}_Y(U)$ such that $nx_1 = nx_2$ for some positive integer n. Replacing U with a cover, assume that x_1 (resp. x_2) lifts to y_1 (resp. y_2) in N. Taking $(*)^{\mathrm{gp}}$ to the left diagram, we have

$$\begin{tikzcd}
\mathbb{Z}[\frac{1}{p}]^n \arrow[r, hook] \arrow[d, two heads, "\beta_p"] & \mathsf{N}^{\mathrm{gp}} \arrow[d, two heads, "\beta"] \\
(\overline{\mathcal{M}}'_Y)^{\mathrm{gp}} \arrow[r, hook] & \overline{\mathcal{M}}^{\mathrm{gp}}_Y.
\end{tikzcd}$$

Since $\overline{\mathcal{M}}_Y$ is saturated and sharp, both $\overline{\mathcal{M}}_Y$ and $\overline{\mathcal{M}}^{\mathrm{gp}}_Y$ are torsion-free. Hence, if $y \in \mathsf{N}^{\mathrm{gp}}$ such that $ny \in \ker \beta_p$, then $y \in \ker \beta$. So $y_1 - y_2 \in \ker \beta$. This implies that $x_1 = x_2$ in $\overline{\mathcal{M}}^{\mathrm{gp}}_Y(U)$ and, *a priori*, in $\overline{\mathcal{M}}_Y(U)$ by integrality. Now we have proved the n-uniqueness of $\overline{\mathcal{M}}_Y$. Other properties follow from the diagram (2.5). ☐

Lemma 2.11. *Let P' be a saturated, sharp, uniquely p-divisible, and p-finitely generated monoid. Then there is an fs sharp submonoid $\mathsf{P}_0 \subset \mathsf{P}'$ such that $\mathsf{P}' = \mathsf{P}_0[\frac{1}{p}]$, where $\mathsf{P}_0[\frac{1}{p}] := \varinjlim_n \frac{\mathsf{P}_0}{p^n} \subset \mathsf{P}'^{,\mathrm{gp}}$.*

Proof. Since P' is sharp and saturated, we embed P' into $\mathsf{P}'^{,\mathrm{gp}} \cong \mathbb{Z}[\frac{1}{p}]^n$ (see the proof of Lemma 2.9). Then, by the definition of being p-finitely generated, there is a set S_0 of $\mathbb{N}[\frac{1}{p}]$-generators of P'. Let $\mathsf{P}_0^{\mathrm{gp}}$ be the $\mathbb{Z}$-module generated by S_0 in $\mathsf{P}'^{,\mathrm{gp}}$, and M_0 be the ($\mathbb{N}$-)monoid generated by S_0 in $\mathsf{P}_0^{\mathrm{gp}} \cap \mathsf{P}'$. Let P_0 be the saturation of M_0 (in $\mathsf{P}_0^{\mathrm{gp}}$); this monoid is saturated, finitely generated by [Ogu18, I. Cor. 2.1.11], and integral by construction.

Note that the saturation P_0 of $\mathsf{M}_0 \subset \mathsf{P}_0^{\mathrm{gp}} \cap \mathsf{P}'$ is in P', as P' is saturated. Hence, P_0 is fine by the above paragraph, and is sharp since it is included in a sharp monoid P'. ☐

Lemma 2.12. *Let $(\mathcal{M}_Y, \alpha)$ be a saturated and strongly fine perfectoid log structure on $Y_{\acute{e}t}$.*

Then, for any geometric point $\overline{x}$ of $Y_{\acute{e}t}$, there is an étale neighborhood $U_{\overline{x}}$ such that $\mathcal{M}_Y$ admits a chart modeled on $\overline{\mathcal{M}}_{Y,\overline{x}}$.

Proof. The proof is similar to that of [DLLZ23a, Prop. 2.3.12-2.3.13], but it is much more involved here, as we do not assume that the log structure is fs. Since this is a local question, we assume that there is an p-finitely generated P together with $\theta : \mathsf{P} \to \mathcal{M}_Y$ factoring through $\mathcal{O}^+_{Y_{\text{ét}}}$, such that $\theta^a : (\mathsf{P}_X)^a \to \mathcal{M}_Y$ is an isomorphism. Note that this implies that $\overline{\theta}_{\overline{x}} : \mathsf{P} \to (\mathsf{P}_X)^a/(\alpha \circ \theta)^{-1}(\mathcal{O}^\times_{Y_{\text{ét}},\overline{x}}) \to \overline{\mathcal{M}}_{Y,\overline{x}}$ is surjective (see [DLLZ23a, Rmk. 2.3.4]). Our goal is reduced to finding a p-finitely generated chart $\mathsf{P}' := \overline{\mathcal{M}}_{Y,\overline{x}}$ in an étale neighborhood of $\overline{x}$.

Firstly, we show that there is a section at the geometric point $\overline{x}$. There is a natural surjective map between integral monoids

$$\mathcal{M}^\flat_{Y,\overline{x}} \to \overline{\mathcal{M}^\flat_{Y,\overline{x}}} = \overline{\mathcal{M}}_{Y,\overline{x}}.$$

The last equality follows from the definition of perfectoid log structures. By Lemma 2.9 above, there is a section from $\overline{\mathcal{M}}_{Y,\overline{x}}$ to $\mathcal{M}^\flat_{Y,\overline{x}}$. Let $\mathsf{P}' := \overline{\mathcal{M}}_{Y,\overline{x}}$ and fix a section $\mathsf{s}^\flat : \mathsf{P}' \to \mathcal{M}^\flat_{Y,\overline{x}}$. Composing $\mathsf{s}^\flat$ with the natural projection $p_0 : \mathcal{M}^\flat_{Y,\overline{x}} \to \mathcal{M}_{Y,\overline{x}}$, we obtain a section

$$\mathsf{s} : \mathsf{P}' \cong (\overline{\mathcal{M}^\flat_Y})_{\overline{x}} \xrightarrow{\mathsf{s}^\flat} \mathcal{M}^\flat_{Y,\overline{x}} \xrightarrow{p_0} \mathcal{M}_{Y,\overline{x}}.$$

Denote by P_0 the fs sharp submonoid in P' as constructed in Lemma 2.11, and denote by S_0' the finite set of generators of P_0. By the proof of both Lemma 2.9 and Lemma 2.11, there is a commutative diagram

$$\begin{tikzcd}
\mathsf{P}_0 \arrow[r, hook] \arrow[d, hook] & \mathsf{P}' \arrow[d, hook] \\
\mathsf{P}_0^{\mathrm{gp}} \arrow[r, hook] & \mathsf{P}'^{,\mathrm{gp}},
\end{tikzcd}$$

where $\mathsf{P}_0^{\mathrm{gp}}$ (resp. $\mathsf{P}'^{,\mathrm{gp}}$) is a finite free $\mathbb{Z}$- (resp. $\mathbb{Z}[\frac{1}{p}]$-) module. Since $\mathcal{M}_Y$ is perfectoid, $\mathcal{M}_{Y,\overline{x}}$ is also perfectoid. Since $\overline{\mathcal{M}^\flat_{Y,\overline{x}}} \to \overline{\mathcal{M}}_{Y,\overline{x}}$ is an isomorphism, we can lift any $s \in \overline{\mathcal{M}}_{Y,\overline{x}}$ to $\widetilde{s} = (s_0, s_1, \dots) \in \mathcal{M}^\flat_{Y,\overline{x}}$ via s. The p^n-th root of $\widetilde{s}$ taken in $\mathcal{M}^\flat_{Y,\overline{x}}$ is given by shifting the limit representing $\widetilde{s}$ to the left by n.

Consider the map $\alpha \circ \mathsf{s} : \mathsf{P}' \to \mathcal{M}^\flat_{Y,\overline{x}} \to \mathcal{M}_{Y,\overline{x}} \xrightarrow{\alpha} \mathcal{O}_{Y,\overline{x}}$. We can make this composition factor through $\mathcal{O}^+_{Y,\overline{x}}$ by adjusting s, which will be explained now. Since Y is perfectoid (and Tate), we have that $\{f \in \mathcal{O}_{Y,\overline{x}} | |f(\overline{x})| > 1\} \subset \mathcal{O}^\times_{Y,\overline{x}}$ and $\{f \in \mathcal{O}_{Y,\overline{x}} | |f(\overline{x})| \leq 1\} = \mathcal{O}^+_{Y,\overline{x}}$. If there is $c \in S_0'$ such that $|\alpha \circ \mathsf{s}(c)(\overline{x})| > 1$, then $f_c := \alpha \circ \mathsf{s}(c) \in \mathcal{O}^\times_{Y,\overline{x}}$. Assume that $|f_c(\overline{x})|$ is maximal among all $c_i \in S_0'$ (noting that $|S_0'|$ is finite). Write $S_0' = \{c_i\}_{i \in I}$. By [Ogu18, I. Cor. 2.2.7] (see also [DLLZ23a, Lem. 2.3.12]), P_0 is embedded in some $\mathbb{N}^m$, where c_i can be uniquely represented by a tuple $\underline{c}_i = (n_1^i, \cdots, n_m^i) \neq (0, 0, \cdots, 0)$ of natural numbers. Denote by $||\underline{c}_i||$ the sum of the coefficients of $\underline{c}_i$ representing c_i; it is not less than 1. Define a map

$$\mathsf{s}^\flat_{/f_c} : \sum_{i \in I} a_i c_i \mapsto \prod_{i \in I} (\mathsf{s}^\flat(c_i) \cdot p_0^{-1}[(\mathsf{s}(c))^{-||\underline{c}_i||}])^{a_i},$$

where $a_i \in \mathbb{N}[\frac{1}{p}]$. The function $(-)^{a_i}$ makes sense, as explained in the last paragraph. Note that $\mathsf{s}^{\flat}_{/f_c}$ is well-defined: On P_0, it is well-defined because $\underline{c}_i$ is uniquely defined; on P', note that any element lies in P_0 by multiplying some p^k and that $\mathcal{M}^{\flat}_{Y,\overline{x}}$ is uniquely p-divisible by the explanation in the last paragraph.

Then $\mathsf{s}^{\flat}_{/f_c} : \mathsf{P}' \to \mathcal{M}^{\flat}_{Y,\overline{x}}$ is a $\mathbb{N}[\frac{1}{p}]$-equivariant morphism. Set $\mathsf{s}_{/f_c} := p_0 \circ \mathsf{s}^{\flat}_{/f_c}$. Then $\alpha \circ \mathsf{s}_{/f_c}$ factors through $\mathcal{O}^{+}_{Y,\overline{x}}$, as desired.

Set $\theta'_{\overline{x}} := \mathsf{s}_{/f_c}$ and $\theta'^{,a}_{\overline{x}} : (\mathsf{P}')^a \to \mathcal{M}_{Y,\overline{x}}$. There is an étale neighborhood U of $\overline{x}$ such that $\theta'_{\overline{x}}$ extends to $\theta' : \mathsf{P}'_U \to \mathcal{M}_Y|_U$, since $\mathsf{s}^{\flat} : \mathsf{P}' \to \mathcal{M}^{\flat}_{Y,\overline{x}}$ extends to an étale neighborhood. Consequently, there is an extension $\theta'^{,a} := (\theta')^a$ for $\theta'^{,a}_{\overline{x}}$ on U. We only need to show that $\theta'^{,a}$ is an isomorphism up to shrinking the étale neighborhood U.

Indeed, $\theta'^{,a}$ is surjective over some $U_1 \in Y_{\text{ét}} \to U$, since $\mathcal{M}_Y|_U/\theta'^{,a}(\mathsf{P}'^{,a}_U) = \theta^a(\mathsf{P}^a_U)/\theta'^{,a}(\mathsf{P}'^{,a}_U)$ is trivial at $\overline{x}$, and both P and P' are uniquely p-divisible and p-finitely generated.

Moreover, this map is injective. The congruence relation R_0 in $\mathsf{P}_0 \times \mathsf{P}_0$ induced by $\mathsf{P}_0 \to \overline{\mathcal{M}}_Y$ is finitely generated by [Ogu18, I. Lem. 2.1.9] and is trivial at $\overline{x}$. Hence, it is trivial over some étale neighborhood U_2 of $\overline{x}$, which we assume there is an étale morphism $U_2 \to U_1$. As both P' and $\overline{\mathcal{M}}_Y$ are integral and uniquely p-divisible, the congruence relation $\mathsf{R} \subset \mathsf{P}' \times \mathsf{P}'$ induced by $\mathsf{P}' \to \overline{\mathcal{M}}_Y$ is trivial over $U_{2,\text{ét}}$. Indeed, for $a, b \in \mathsf{P}'$, if $a \sim b$ in $\overline{\mathcal{M}}_Y$ over $U_{2,\text{ét}}$, so is $p^i a$ and $p^i b$. Conversely, we can divide relations in R_0 by p: if $a \sim b$, let $a' = \frac{a}{p}, b' = \frac{b}{p}$ in P', then the images c_1, c_2 of a' and b' in $\overline{\mathcal{M}}_Y$ are equal since $pc_1 = pc_2$ by the unique p-divisibility of $\overline{\mathcal{M}}_Y$. So R is trivial if and only if R_0 is trivial. This completes the proof. ☐

2.1.5. We define log diamonds for the cases in §2.1.3.

In Case 1,

Definition 2.13. *Let $(X, \mathcal{M}_X, \alpha)$ be an (étale sheafy) fs log adic space over* $\operatorname{Spa} \mathbb{Z}_p$*. Then the log diamond* $(X, \mathcal{M}_X)^{\Diamond}$ *(or* $X^{\log \Diamond}$*) is a functor sending* $S = \operatorname{Spa}(A, A^+) \in \operatorname{Ob} \operatorname{Perf}$ *to the isomorphism classes* $\{(S^{\sharp}, \iota, \mathcal{M}_{S^{\sharp}}, f)\}/\simeq$*, where* $(S^{\sharp}, \iota)$ *denotes an untilt as in §2.1.1,* $\mathcal{M}_{S^{\sharp}}$ *is a saturated and fine perfectoid log structure, and* $f : S^{\sharp} \to X$ *is a morphism between adic spaces that induces a morphism between log structures* $f^*\mathcal{M}_X \to \mathcal{M}_{S^{\sharp}}$*.*

In Case 2, we can also define small and big log diamonds.

Definition 2.14. *Let* $(X, \mathcal{M}_X)$ *be an fs log scheme over* $\mathbb{Z}_p$*. We define a* ***log big diamond*** *functor* $(X, \mathcal{M}_X)^{\Diamond}$ *(or* $X^{\log \Diamond}$*) as*

$$S = \operatorname{Spa}(A, A^+) \in \operatorname{Perf} \mapsto \{(S^{\sharp} = \operatorname{Spa}(A^{\sharp}, A^{\sharp,+}), \iota, \mathcal{M}_{S^{\sharp}}, f)\}/\simeq,$$

where all but the last term in the tuple are as above, and $f : \operatorname{Spa}(A^{\sharp}, A^{\sharp,+}) \to \operatorname{Spec} A^{\sharp} \to X$ *induces a morphism* $f^*\mathcal{M}_X \to \mathcal{M}_{S^{\sharp}}$*; note that* f *induces a morphism between étale sites* $S^{\sharp}_{\text{ét}} \to (\operatorname{Spec} A^{\sharp})_{\text{ét}} \to X_{\text{ét}}$*, and that all perfectoid spaces are étale sheafy by* [DLLZ23a, Cor. A.11]*.*

We define a ***log small diamond*** *functor* $(X, \mathcal{M}_X)^{\diamond}$ *(or* $X^{\log \diamond}$*) in a similarly-written form* $S \in \operatorname{Perf} \mapsto \{(S^{\sharp} = \operatorname{Spa}(A^{\sharp}, A^{\sharp,+}), \iota, \mathcal{M}_{S^{\sharp}}, f)\}/\simeq$*, but here* f *is a morphism* $f : \operatorname{Spa}(A^{\sharp}, A^{\sharp,+}) \to \operatorname{Spec} A^{\sharp,+} \to X$*.*

In Case 3, we can associate a log diamond functor; we denote it again by $(X, \mathcal{M}_X)^{\Diamond}$ or $X^{\log \Diamond}$.

Definition 2.15. *Let* $(X, \mathcal{M}_X)$ *be an fs log formal scheme over* $\operatorname{Spf} \mathbb{Z}_p$*. The functor* $(X, \mathcal{M}_X)^{\Diamond}$ *sends* $S = \operatorname{Spa}(A, A^+) \in \operatorname{Perf}$ *to*

$$\{(S^{\sharp} = \operatorname{Spa}(A^{\sharp}, A^{\sharp,+}), \iota, \mathcal{M}_{S^{\sharp}}, f)\}/\simeq,$$

where $f : S^{\sharp} \to X$ *is a morphism that induces* $f^*\mathcal{M}_X \to \mathcal{M}_{S^{\sharp}}$ *and other components are the same as above.*

This symbol is compatible with the log small diamond for Case 2: Let $\mathrm{Spf}(A, I)$ be an affine formally of finite type formal scheme. Suppose that A is also p-adic complete. Let $X = \operatorname{Spec} A$; and we equip X with an fs log structure $\mathcal{M}$. Then $X^{\log\diamond}$ is by definition the same as $(\mathrm{Spf}(A, (p)), \mathcal{M})^{\lozenge}$; here the latter $\mathcal{M}$ is the log structure pulled back from $X_{\text{ét}}$.

In the case of (bounded) fs log p-adic formal schemes, Definition 2.15 essentially comes from the category $(X, \mathcal{M}_X)^{\text{perf}}_{\mathbb{A}}$ in [KY25, Def. 7.34 and Thm. 7.35].

Remark 2.16 (cf. Remark 2.8)**.** *As we can see later from Lemma 2.37, one might remove the p-finite generation condition in the definition of log diamonds given above. But it is this more restrictive class of log structures that is useful to our purpose.*

Assume that X is a $\mathbb{Z}_p$-scheme satisfying (SF). The diamond functors in Case 2 also admit constructions via the diamond functor in the log adic space case.

Lemma 2.17. *Under the assumption above, the log big diamond functor can be constructed as a functor sending any affinoid perfectoid $S = \mathrm{Spa}(A, A^+) \in \operatorname{Ob}\mathrm{Perf}$ to the isomorphism classes*

$$\{(S^\sharp, \iota, \mathcal{M}_{S^\sharp}, f)\}/\simeq,$$

where $f : (S^\sharp, \mathcal{M}_{S^\sharp}) \to (X^{\text{ad}}, \nu^{\text{ad},-1}\mathcal{M}_X)$ a morphism. More precisely, f is a morphism between adic spaces $f : S^\sharp \to X^{\text{ad}}$ that induces a morphism of sites

$$S^\sharp_{\acute{e}t} \xrightarrow{f} X^{\text{ad}}_{\acute{e}t} \xrightarrow{\nu^{\text{ad}}} X_{\acute{e}t}$$

together with a morphism $f^{-1} \circ \nu^{\text{ad},-1}\mathcal{M}_X \to \mathcal{M}_{S^\sharp}$.

Similarly, the log small diamond functor can be constructed as a functor the same as above, but changing f to $f : (S^\sharp, \mathcal{M}_{S^\sharp}) \to (\widehat{X}^{\text{ad}}, \widehat{\nu}^{-1}\mathcal{M}_X)$, which, more precisely, is a morphism of adic spaces $f : S^\sharp \to \widehat{X}^{\text{ad}}$ that induces a morphism of sites:

$$S^\sharp_{\acute{e}t} \xrightarrow{f} \widehat{X}^{\text{ad}}_{\acute{e}t} \xrightarrow{\widehat{\nu}} X_{\acute{e}t}$$

together with a morphism $f^{-1} \circ \widehat{\nu}^{-1}\mathcal{M}_X \to \mathcal{M}_{S^\sharp}$.

Proof. The fact that one can construct diamonds via X^{ad} and $\widehat{X}^{\text{ad}}$ is in [AGLR22, §2.2]. By [DLLZ23a, Cor. A.11], X^{ad} and $\widehat{X}^{\text{ad}}$ are étale sheafy. $\square$

The main theorem of §2.1 is the following:

Theorem 2.18. *The functors $X^{\log\lozenge}$ and $X^{\log\diamond}$ in Definition 2.13, Definition 2.14 and Definition 2.15 are v-sheaves over* Perf.

We will complete the proof of this theorem in §2.1.6 and §2.1.7.

2.1.6. Let Y be a perfectoid space. Let $\nu : Y_v \to Y_{\text{ét}}$ be the natural projection from the v-site to the étale site of Y. There is a pullback functor $\nu^{-1} : Y^\sim_{\text{ét}} \to Y^\sim_v$ and a pushforward functor $\nu_* : Y^\sim_v \to Y^\sim_{\text{ét}}$ between topoi.

Let $\mathcal{LOG}_?$ (resp. $\mathcal{LOG}^{\text{int}}_?$) be the category (resp. the full subcategory) of log structures (resp. integral log structures) on $Y_?$ for $? = \text{ét}$ and v. Let $\mathcal{M}$ be a log structure on $Y_{\text{ét}}$. Define the pullback of $\mathcal{M}$ to Y_v as $\nu^*\mathcal{M} := (\nu^{-1}\mathcal{M})^a$.

The following statement can be viewed as an analogue of [Ols03, App. A] and [Ogu18, III. Prop. 1.4.1] for perfectoid spaces.

Lemma 2.19. *With the conventions above, the pullback $\nu^* : \mathcal{LOG}_{Y_{\acute{e}t}} \to \mathcal{LOG}_{Y_v}$ restricts to a functor from $\mathcal{LOG}^{\text{int}}_{Y_{\acute{e}t}}$ to $\mathcal{LOG}^{\text{int}}_{Y_v}$, and this restriction induces an equivalence from the category of fine perfectoid log structures on $Y_{\acute{e}t}$ to the category of fine perfectoid log structures on Y_v.*

Moreover, it induces an equivalence between the category of saturated and fine perfectoid log structures on $Y_{\acute{e}t}$ and the category of saturated and fine perfectoid log structures on Y_v.

We will show Lemma 2.19 in the rest of §2.1.6. The procedures are similar to the two references above.

We first show the full faithfulness of $\nu^*|_{\mathcal{LOG}^{\mathrm{int}}_{Y_{\text{ét}}}}$ and show that the essential image of this functor lies in $\mathcal{LOG}^{\mathrm{int}}_{Y_v}$.

Lemma 2.20. *The functor $\nu^* : \mathcal{LOG}^{\mathrm{int}}_{Y_{\acute{e}t}} \to \mathcal{LOG}_{Y_v}$ is fully faithful. That is, we have*

$$\mathrm{Hom}_{\mathcal{LOG}_{Y_{\acute{e}t}}}(\mathcal{L}_1, \mathcal{L}_2) \cong \mathrm{Hom}_{\mathcal{LOG}_{Y_v}}(\nu^*\mathcal{L}_1, \nu^*\mathcal{L}_2)$$

for integral log structures $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{LOG}^{\mathrm{int}}_{Y_{\acute{e}t}}$.

Proof. Denote by $\mathcal{LOG}^{\mathrm{pre}}_{Y_?}$ the category of prelog structures on the ? topology of Y. We have $\mathrm{Hom}_{\mathcal{LOG}_{Y_v}}(\nu^*\mathcal{L}_1, \nu^*\mathcal{L}_2) \cong \mathrm{Hom}_{\mathcal{LOG}^{\mathrm{pre}}_{Y_v}}(\nu^{-1}\mathcal{L}_1, \nu^*\mathcal{L}_2) \cong \mathrm{Hom}_{\mathcal{LOG}^{\mathrm{pre}}_{Y_{\text{ét}}}}(\mathcal{L}_1, \nu_*\nu^*\mathcal{L}_2)$ by adjunctions. The last one is isomorphic to $\mathrm{Hom}_{\mathcal{LOG}_{Y_{\text{ét}}}}(\mathcal{L}_1, \nu_*\nu^*\mathcal{L}_2)$. It suffices to show that the last term is canonically isomorphic to $\mathcal{L}_2$, which we will do in Lemma 2.21. ☐

Lemma 2.21. *The canonical morphism between log structures $\mathcal{L} \to \nu_*\nu^*\mathcal{L}$ is an isomorphism for any $\mathcal{L} \in \mathcal{LOG}^{\mathrm{int}}_{Y_{\acute{e}t}}$.*

Proof. There is a functor defined on Y_v by

$$(f : Y' \to Y) \in Y_v \mapsto f^*\mathcal{L} := (f^{-1}\mathcal{L})^a \in \mathcal{LOG}_{Y'_{\text{ét}}}.$$

We claim that the presheaf

$$(f : Y' \to Y) \mapsto f^*\mathcal{L}(Y')$$

is a sheaf on Y_v. If we know this, then $\nu^*\mathcal{L}$ is this (pre)sheaf, and the desired result follows from the definition of ν_*.

We now show the claim in the last paragraph. Let $\mathcal{L}' := f^*\mathcal{L}$ and $\mathcal{L}'' = p_1^*f^*\mathcal{L} = p_2^*f^*\mathcal{L}$. We show that the composition

$$\begin{tikzcd}
\mathcal{L}(Y) \arrow[r] & \mathrm{eq}(\mathcal{L}'(Y') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & \mathcal{L}''(Y' \times_Y Y')).
\end{tikzcd}$$

is an isomorphism.

First of all, $f^*\mathcal{L}$ is integral: By [DLLZ23a, Lem. 2.2.4] and [Ogu18, I. Prop. 1.3.4], we check pointwise that $f^{-1}\mathcal{L}$ is integral and $f^*\mathcal{L}$ is integral.

Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}^\times(Y) \arrow[r] \arrow[d] & \mathrm{eq}(\mathcal{O}^\times(Y') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] \arrow[d] & \mathcal{O}^\times(Y' \times_Y Y')) \arrow[d] \\
\mathcal{L}(Y) \arrow[r] \arrow[d] & \mathrm{eq}(\mathcal{L}'(Y') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] \arrow[d] & \mathcal{L}''(Y' \times_Y Y')) \arrow[d] \\
\overline{\mathcal{L}}(Y) \arrow[r] & \mathrm{eq}(\overline{\mathcal{L}}'(Y') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & \overline{\mathcal{L}}''(Y' \times_Y Y')).
\end{tikzcd} \tag{2.6}$$

It follows from [Sch26, Thm. 8.7] that $\mathcal{O}_Y$ is a v-sheaf, so is $\mathcal{O}_Y^\times$. We have the isomorphism of the first row of (2.6). The last row is also an isomorphism. This follows from [Sch26, Prop. 14.7 and 14.8].

We now examine the middle row of (2.6). Then it follows from a diagram chasing that $\mathcal{L}(V) \to \mathcal{L}'(V \times_Y Y')$ is injective for any $V \in Y_{\text{ét}}$: Indeed, suppose that there are sections $s_1, s_2 \in \mathcal{L}(V)$ mapping to the same element $s \in \mathcal{L}'(V \times_Y Y')$. We have that $s_1 = s_2 + u$ for $u \in \mathcal{O}^\times(V)$. Then $s = s + u$ in $\mathcal{L}'(V \times_Y Y')$, and therefore $u = 0$ in $\mathcal{L}'(V \times_Y Y')$ by integrality. Then $u = 0$ in $\mathcal{O}^\times(V) \to \mathcal{L}(V)$ by the paragraph above, as desired.

On the other hand, for any $s \in \mathrm{eq}(\mathcal{L}'(Y') \rightrightarrows \mathcal{L}''(Y' \times_Y Y'))$, there is étale locally $s' \in \mathcal{L}(U)$ for some $U \in Y_{\text{ét}}$ mapping to $s \in \mathcal{L}'(Y' \times_Y U)$ by diagram chasing. The element s' on the étale cover U of Y glues to a section on Y since we can check it over the cover $Y' \times_Y U$ by the injectivity we just proved. We now have the isomorphism of the middle arrow, as desired. ☐

In the proof, we have shown that

Corollary 2.22. *The pullback functor ν^* restricts to a fully faithful functor*

$$\nu^* : \mathcal{LOG}^{\mathrm{int}}_{Y_{\acute{e}t}} \longrightarrow \mathcal{LOG}^{\mathrm{int}}_{Y_v}.$$

Corollary 2.23. *The functor $\nu^* : \mathcal{LOG}^{\mathrm{int}}_{Y_{\acute{e}t}} \longrightarrow \mathcal{LOG}^{\mathrm{int}}_{Y_v}$ sends fine perfectoid objects to fine perfectoid objects.*

Proof. All conditions we need follow from the construction in Lemma 2.21. ☐

We then show the essential surjectivity. Fix $\mathcal{M} \in \mathcal{LOG}_{Y_v}$ that is fine perfectoid. We show that

Lemma 2.24. *The adjunction morphism $\nu^*\nu_*\mathcal{M} \to \mathcal{M}$ is an isomorphism.*

Proof. There is an exact sequence $1 \to \mathcal{O}^{\times}_{Y_{\text{ét}}} \to \nu_*\mathcal{M} \to \overline{\nu_*\mathcal{M}} \to 0$. Applying ν^{-1}, we get $1 \to \nu^{-1}\mathcal{O}^{\times}_{Y_{\text{ét}}} \to \nu^{-1}\nu_*\mathcal{M} \to \nu^{-1}\overline{\nu_*\mathcal{M}} \to 0$. Moreover, we have a commutative diagram of exact sequences

(2.7)

$$\begin{tikzcd}
1 \arrow[r] & \nu^*\mathcal{O}^{\times}_{Y_{\text{ét}}} \arrow[r] \arrow[d, "\simeq"] & \nu^*\nu_*\mathcal{M} \arrow[r] \arrow[d] & \nu^{-1}\overline{\nu_*\mathcal{M}} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathcal{O}^{\times}_{Y_v} \arrow[r] & \mathcal{M} \arrow[r] & \overline{\mathcal{M}} \arrow[r] & 0.
\end{tikzcd}$$

We first show that the composition

$$\nu^{-1}\overline{\nu_*\mathcal{M}} \to \nu^{-1}\nu_*\overline{\mathcal{M}} \to \overline{\mathcal{M}} \tag{2.8}$$

is injective. By the proof of Lemma 2.21, for any v-cover $(f : X \to Y) \in Y_v$, $\nu^*\nu_*\mathcal{M}(X)$ (resp. $\nu^{-1}\overline{\nu_*\mathcal{M}}(X)$) is given by $f^*\nu_*\mathcal{M}(X)$ (resp. $f^{-1}\overline{\nu_*\mathcal{M}}(X)$) with a surjection $f^*\nu_*\mathcal{M} \to f^{-1}\overline{\nu_*\mathcal{M}}$ obtained by pulling back $\nu_*\mathcal{M} \to \overline{\nu_*\mathcal{M}}$. Fix such a morphism f. Suppose that there are $\overline{s}_1, \overline{s}_2 \in f^{-1}\overline{\nu_*\mathcal{M}}(X)$ mapping to the same element $\overline{s} \in \overline{\mathcal{M}}(X)$. Up to replacing X with an étale cover, there is a cover $U \in X_{\text{ét}}$ that is the pullback of an étale cover U_Y of Y via f, such that $\overline{s}_1, \overline{s}_2 \in \overline{\nu_*\mathcal{M}}(U_Y) = \nu_*\overline{\mathcal{M}}(U_Y) = \overline{\mathcal{M}}(U_Y)$. The equation $\overline{\nu_*\mathcal{M}} = \nu_*\overline{\mathcal{M}}$ is proved in Lemma 2.25 below. Moreover, they lift to $s_1, s_2 \in \mathcal{M}(U'_Y)$ for some cover $U'_Y \in U_{Y,\text{ét}}$, respectively. On the other hand, there is a v-cover $V \to U' := U'_Y \times_Y X$ with $u \in \mathcal{O}^{\times}(V)$ such that $s_1 = s_2 + u$ in $\mathcal{M}(V)$. The descent data of s_1 and s_2 from V to U', along with the integrality of $\mathcal{M}$, induces a descent datum of u from V to U'. So $u \in \mathcal{O}^{\times}(U')$ and $\overline{s}_1 = \overline{s}_2$ in $\overline{\mathcal{M}}(U')$. We have shown the desired claim since $U' \to U_Y$ is a v-cover.

A similar argument as above implies that $\nu^*\nu_*\mathcal{M} \to \mathcal{M}$ is injective. For $Y' \in Y_{\text{ét}}$ and $s_1, s_2 \in \nu^*\nu_*\mathcal{M}(Y')$ mapping to $s \in \mathcal{M}(Y')$. By diagram chasing, s_1 and s_2 project to the same $\overline{s} \in \nu^{-1}\overline{\nu_*\mathcal{M}}(Y')$ and $s_1 = s_2 + u'$ in $\nu^*\nu_*\mathcal{M}(Y'')$, where Y'' is a v-cover of Y' and $u \in \mathcal{O}^{\times}(Y'')$. By the integrality of $\nu^*\nu_*\mathcal{M}$, u descends to $\mathcal{O}^{\times}(Y')$. Mapping to $\mathcal{M}(Y')$, by the integrality of $\mathcal{M}$ and $\nu^*\mathcal{O}^{\times}_{Y_{\text{ét}}} = \mathcal{O}^{\times}_{Y_v}$, we have that $u = 0$, as desired.

We next show that (2.8) is an isomorphism. In fact, by assumption, for any geometric point $\overline{x} \in X_{\text{ét}}$, there is an étale neighborhood $U_{\overline{x}}$ of it such that $\mathcal{M}$ admits a chart modeled on a weakly p-finitely generated P denoted by $\theta : \mathsf{P}_{U_{\overline{x}}} \to \mathcal{M}|_{U_{\overline{x}}}$, and such that θ factors through $\nu^*\nu_*\mathcal{M}|_{U_{\overline{x}}}$. Then there is a morphism $\mathsf{P} \to \nu^{-1}\overline{\nu_*\mathcal{M}} \to \nu^{-1}\nu_*\overline{\mathcal{M}} \to \overline{\mathcal{M}}$ over $U_{\overline{x}}$ whose composition is surjective. Then (2.8) is surjective, as desired.

Finally, the remainder of the lemma follows from a diagram chasing for (2.7). ☐

In fact, we have

Lemma 2.25. *The natural morphism $\overline{\nu_*\mathcal{M}} \to \nu_*\overline{\mathcal{M}}$ is an isomorphism.*

Proof. It is an injection, as there is a commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}^\times_{Y_{\text{ét}}} \arrow[r] \arrow[d, "\simeq"] & \nu_*\mathcal{M} \arrow[r] \arrow[d] & \overline{\nu_*\mathcal{M}} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \nu_*\mathcal{O}^\times_{Y_v} \arrow[r] & \nu_*\mathcal{M} \arrow[r] & \nu_*\overline{\mathcal{M}} &
\end{tikzcd}$$

and the injectivity is proved by diagram chasing. We show that this is a surjection. It can be checked that, for $U \in Y_{\text{ét}}$, $\overline{\nu_*\mathcal{M}}(U) =$

$$\{s \in \overline{\mathcal{M}}(U) | s \text{ can be lifted to } s' \in \mathcal{M}(V) \text{ for some étale cover } V \to U\}.$$

Fix $s \in \overline{\mathcal{M}}(U)$ for $U \in Y_{\text{ét}}$. The functor $\overline{\mathcal{M}}_s$ sending $V \in U_v$ to $\{s' \in \mathcal{M}(V)|s' \mapsto s|_V\}$ is a $\mathbb{G}_m$-torsor on U_v. By [SW20, Lem. 17.1.8] and [KL19, Thm. 2.5.8], $\overline{\mathcal{M}}_s$ is the pullback of an étale $\mathbb{G}_m$-torsor $\overline{\mathcal{N}}_s = \nu_*\overline{\mathcal{M}}_s$. So $\overline{\mathcal{N}}_s$ étale locally has a section, and s étale locally admits a lifting. ☐

We now show the second paragraph of Lemma 2.19.

Lemma 2.26. *Let $\mathcal{L} \in \mathcal{LOG}^{\text{int}}_{Y_{\text{ét}}}$. Then $\mathcal{L}$ is fine perfectoid if and only if $\nu^*\mathcal{L}$ is. In this case, $\mathcal{L}$ is saturated if and only if $\nu^*\mathcal{L}$ is so.*

Proof. For fine perfectoidness, the "only if" part is Corollary 2.23. For the "only if" part of saturatedness, if $\mathcal{L}$ is saturated, then, by the construction in Lemma 2.21, we see that $f^{-1}\overline{\mathcal{L}}$ is saturated by [DLLZ23a, Lem. 2.2.4], and $f^*\mathcal{L}$ is saturated by [Ogu18, I. Prop. 1.3.5] and [DLLZ23a, Lem. 2.2.4].

The "if" parts are immediate from Lemma 2.21 and the definitions. ☐

This completes the proof of Lemma 2.19.

2.1.7. *End of the proof.* From Lemma 2.19, we obtain the following proposition:

Proposition 2.27. *Let $(Y, \mathcal{M}_Y)$ be a log perfectoid space (cf. [DLLZ23a, Def. 2.2.2(9)]) that is fs. Then the v-descent data of the functor sending $(Y', f : Y' \to Y) \in \text{Perfd}_{/Y}$ to the groupoid*

$$\{\textit{saturated and fine perfectoid log structures } \mathcal{M} \textit{ on } Y'_{\acute{e}t} \textit{ with } f^*\mathcal{M}_Y \to \mathcal{M}\}$$

are effective.

Proof. By Lemma 2.19, the functor in the proposition is isomorphic to the functor sending $(Y', f : Y' \to Y) \in \text{Perfd}_{/Y}$ to the groupoid

$$\{\text{saturated and fine perfectoid log structures } \mathcal{M} \text{ on } Y'_v \text{ with } \nu^*f^*\mathcal{M}_Y \to \mathcal{M}\}.$$

Let $h : Y'' \to Y'$ be a v-cover. Let $(\mathcal{M}'', \alpha'')$ be a saturated and fine perfectoid log structure on Y''_v equipped with a descent datum $\sigma : p_1^*\mathcal{M}'' \simeq p_2^*\mathcal{M}''$ on $Y'' \times_{Y'} Y''$ which satisfies the following conditions:

- $p_2^*\alpha'' \circ \sigma = \sigma_0 \circ p_1^*\alpha''$, where σ_0 is the canonical descent datum associated with pulling back $\mathcal{O}_{Y'}$ to Y'' via h.
- There is a morphism $h^\sharp : \nu^*h^*f^*\mathcal{M}_Y \to \mathcal{M}''$ such that $(p_2^*h^\sharp) \circ \sigma_Y = \sigma \circ (p_1^*h^\sharp)$, where σ_Y is the canonical descent datum of $\nu^*h^*f^*\mathcal{M}_Y = h^*\nu^*f^*\mathcal{M}_Y$ induced by pulling back $\nu^*f^*\mathcal{M}_Y$ via h.

This descent datum determines and is determined by a log structure $(\mathcal{M}', \alpha')$ on Y'_v together with a morphism $\nu^*f^*\mathcal{M}_Y \to \mathcal{M}'$. This log structure $\mathcal{M}'$ is saturated and fine perfectoid. Taking ν_*, we get the desired morphism $f^*\mathcal{M}_Y \to \nu_*\mathcal{M}'$ (cf. Lemma 2.21). Then the Proposition follows from Lemma 2.19. ☐

We then have the following corollary:

Corollary 2.28. *Let $(X, \mathcal{M}_X)$ be one of the Cases 1-3.*
Then the functor "$\mathrm{Hom}(-, (X, \mathcal{M}_X))$*" on* Perfd *defined as follows is a v-sheaf:*

$$Y \in \mathrm{Perfd} \longmapsto \{f : (Y, \mathcal{M}) \to (X, \mathcal{M}_X) | \mathcal{M} \textit{ is saturated and fine perfectoid}\}/\simeq .$$

Proof. By [SW20, Thm. 17.1.3 and Thm. 18.1.1], $\mathrm{Hom}(-, X)$ is a v-sheaf on Perfd. We then get the desired result by applying Proposition 2.27 where $\mathcal{M}_Y = f^*\mathcal{M}_X$. ☐

Proof of Theorem 2.18. Now we can show Theorem 2.18 with these facts. Indeed, since Untilt is a v-sheaf by [Sch26, Lem. 15.1], we only have to show that $\mathrm{Hom}(-, (X, \mathcal{M}_X))$ is a v-sheaf on Perfd. But this is Corollary 2.28. ☐

2.1.8. Suppose that X is of finite type over $\mathbb{Z}_p$ and satisfies (SF). It follows from the definition that there is a natural "structural morphism" $X^{\log\diamond} \to \mathrm{Spd}\,\mathbb{Z}_p$. Moreover,

Lemma 2.29. *There is a natural injective morphism between v-sheaves*

$$X^{\log\diamond} \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p \longrightarrow (X_{\mathbb{Q}_p})^{\log\Diamond}.$$

Proof. For any $S := \mathrm{Spa}(A, A^+) \in \mathrm{Perf}$, $X^{\log\diamond} \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p(S)$ parametrizes isomorphism classes of $\{S^\sharp = \mathrm{Spa}(A^\sharp, A^{\sharp,+}), f : (S^\sharp, \mathcal{M}_{S^\sharp}) \to (\widehat{X}^{\mathrm{ad}}, \widehat{\nu}_X^{-1}\mathcal{M}_X)\}$ where $A^\sharp$'s are $\mathbb{Q}_p$-algebras. To define a class in $(X_{\mathbb{Q}_p})^{\log\Diamond}(S)$, it remains to define a morphism $f_{\mathbb{Q}_p} : (S^\sharp, \mathcal{M}_{S^\sharp}) \to ((X_{\mathbb{Q}_p})^{\mathrm{ad}}, \nu_{X_{\mathbb{Q}_p}}^{\mathrm{ad},-1}\mathcal{M}_{X_{\mathbb{Q}_p}})$. For the morphism $f_{\mathbb{Q}_p} : S^\sharp \to (X_{\mathbb{Q}_p})^{\mathrm{ad}}$, it is defined as $S^\sharp \xrightarrow{f} \widehat{X}_\eta^{\mathrm{ad}} \hookrightarrow (X_{\mathbb{Q}_p})^{\mathrm{ad}}$; for the morphism between sheaves $\nu_{X_{\mathbb{Q}_p}}^{\mathrm{ad},-1}\mathcal{M}_{X_{\mathbb{Q}_p}} \to \mathcal{M}_{S^\sharp}$, note that there is a commutative diagram of sites

$$\begin{tikzcd}
(\widehat{X}_\eta^{\mathrm{ad}})_{\text{ét}} \arrow[r, "i_1"] \arrow[d, "j_1"] \arrow[dr, "\widehat{\nu}_{X_{\mathbb{Q}_p}}"] & (\widehat{X}^{\mathrm{ad}})_{\text{ét}} \arrow[d, "j_2"] \arrow[dr, "\widehat{\nu}_X"] & \\
((X_{\mathbb{Q}_p})^{\mathrm{ad}})_{\text{ét}} \arrow[r, "i_2"] \arrow[dr, dashed, "\nu_{X_{\mathbb{Q}_p}}^{\mathrm{ad}}"'] & (X^{\mathrm{ad}})_{\text{ét}} \arrow[dr, dashed, "\nu_X^{\mathrm{ad}}"'] & \\
& X_{\mathbb{Q}_p,\text{ét}} \arrow[r, "i_3"] & X_{\text{ét}}.
\end{tikzcd}$$

For a fixed $f : S^\sharp \to \widehat{X}_\eta^{\mathrm{ad}}$, a morphism $f^{-1}i_1^{-1}\widehat{\nu}_X^{-1}\mathcal{M}_X \to \mathcal{M}_{S^\sharp}$ corresponds to a morphism $\alpha_0 : f^{-1}j_1^{-1}\nu_{X_{\mathbb{Q}_p}}^{\mathrm{ad},-1}i_3^{-1}\mathcal{M}_X \to \mathcal{M}_{S^\sharp}$. The latter one induces a morphism $\alpha : f^{-1}j_1^{-1}\nu_{X_{\mathbb{Q}_p}}^{\mathrm{ad},-1}i_3^*\mathcal{M}_X \to \mathcal{M}_{S^\sharp}$, where $i_3^*\mathcal{M}_X = \mathcal{M}_{X_{\mathbb{Q}_p}}$. Conversely, α also uniquely determines α_0. So the injectivity follows from the injectivity of $X^\diamond \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p \to X_{\mathbb{Q}_p}^\Diamond$. ☐

Definition 2.30. *Let $(X, \mathcal{M}_X)$ be an fs log scheme over* $\mathrm{Spec}\,\mathbb{Z}_p$. *Define the slashed log diamond $(X, \mathcal{M}_X)^{\Diamond/}$ (or denoted by $X^{\log\Diamond/}$) as the quotient v-sheaf*

$$(X, \mathcal{M}_X)^{\Diamond/} := X^{\log\diamond} \coprod_{X^{\log\diamond}\times_{\mathrm{Spd}\,\mathbb{Z}_p}\mathrm{Spd}\,\mathbb{Q}_p} (X_{\mathbb{Q}_p})^{\log\Diamond}.$$

Note that the definition makes sense by Theorem 2.18.

Lemma 2.31. *When X is proper over $\mathbb{Z}_p$, the natural injections $X^{\log\diamond} \hookrightarrow X^{\log\Diamond/} \hookrightarrow X^{\log\Diamond}$ are isomorphisms.*

Proof. This follows from the construction and the fact that $X^{\mathrm{ad}} \cong \widehat{X}^{\mathrm{ad}}$ when X is proper over $\mathbb{Z}_p$. ☐

Let us also remark that there is another way of defining a log diamond, which is a generalization of [SW20, Def. 8.3.1].

Remark 2.32. *Let* FSPerf *be the category of saturated and fine perfectoid log perfectoid spaces of characteristic* p*. We then define a log diamond* X *as a pro-étale sheaf on* FSPerf *such that: (1) There is a surjective map between sheaves* $Y \to X$*, where* Y *is representable by an object in* FSPerf*; (2) the fiber product of sheaves "*$Y \times_X Y$*" is also representable by an object in* FSPerf*, denoted by* $Y \times_X^{sat} Y$*, such that the two projections* $p_1,\, p_2 : Y \times_X^{sat} Y \to Y$ *are (non-log) pro-étale.*

We expect that Definition 2.13 should satisfy this definition when X *is analytic over* $\operatorname{Spa} \mathbb{Z}_p$ *(cf. Lemma 2.47). We do not explore this direction in the current paper.*

One can also change the definition of log diamonds by changing the category FSPerf *to other categories with a different class of log structures. This definition may also be viewed as an analogue of the definition of a log algebraic space in the second sense due to Kajiwara-Kato-Nakayama (see* [KKN15, 10.1]*) in perfectoid geometry. The definition of log diamonds in* [KY25, Def. 7.3] *may also be viewed as an analogue of log algebraic spaces in the first sense in* [KKN15] *for some category of log perfectoid spaces.*

2.2. **Log shtukas.** We define a notion of p-adic shtukas in log geometry.

Let $(X, \mathcal{M}_X)$ be one of the cases in §2.1.3.

2.2.1. As §1.3, let G be a connected linear algebraic group over $\mathbb{Q}$ that has a quasi-parahoric model $\mathcal{G}$ over $\mathbb{Z}_p$ in the sense of Definition 1.18. Note that we still use the symbol G and $\mathcal{G}$ here, but G might not be reductive.

Let $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ be a minuscule cocharacter in the sense of Definition 1.11 (2), and denote by $\{\mu\}$ the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of it. Let E be the field of definition of $\{\mu\}$.

In what follows, ? denotes $\lozenge$ or $\diamond$. Since $(X, \mathcal{M}_X)^?$ is a functor from Perf to isomorphism classes, we can and will regard $(X, \mathcal{M}_X)^?$ as a functor $(X, \mathcal{M}_X)^? : \mathrm{Perf}^{\mathrm{op}} \to \mathsf{Categories}$ that sends $S \in \mathrm{Perf}$ to the collection of tuples $(S^\sharp, \mathcal{M}_{S^\sharp}, \alpha_{S^\sharp}, f)$ viewed as a category, which is still denoted by $(X, \mathcal{M}_X)^?(S)$. More precisely, the objects of $(X, \mathcal{M}_X)^?(S)$ are those $(S^\sharp, \mathcal{M}_{S^\sharp}, \alpha_{S^\sharp}, f)$, while the morphisms between $(S_1^\sharp, \mathcal{M}_{S_1^\sharp}, \alpha_{S_1^\sharp}, f_1)$ and $(S_2^\sharp, \mathcal{M}_{S_2^\sharp}, \alpha_{S_2^\sharp}, f_2)$ are the morphisms between log adic spaces

$$g : (S_1^\sharp, \mathcal{M}_{S_1^\sharp}, \alpha_{S_1^\sharp}) \to (S_2^\sharp, \mathcal{M}_{S_2^\sharp}, \alpha_{S_2^\sharp})$$

such that $f_2 \circ g = f_1$.

By Lemma 2.19, Theorem 2.18 and Proposition 2.27, this functor is a v-stack, which we abusively denote by

$$p^? : (X, \mathcal{M}_X)^? \to \mathrm{Perf}\,.$$

2.2.2. On the other hand, let

$$\mathrm{Sht}_{\mathcal{G},\mu} : \mathrm{Perf}^{\mathrm{op}} \to \mathsf{Groupoids}$$

be the v-stack over Perf sending $S \in \mathrm{Perf}$ to the groupoid with objects $(S^\sharp; (\mathscr{P}, \phi_{\mathscr{P}}))$, where $S^\sharp$ is an untilt of S and $(\mathscr{P}, \phi_{\mathscr{P}})$ is a $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$ bounded by μ (see [PR24, Def. 2.4.3]). For a fixed $S^\sharp$, let $\mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp)$ be the groupoid of $\mathcal{G}$-shtukas over S with one leg at $S^\sharp$ bounded by μ.

Write the fibered category in groupoids corresponding to $\mathrm{Sht}_{\mathcal{G},\mu}$ by $p_{\mathrm{Sht}} : \mathcal{SHT}_{\mathcal{G},\mu} \to \mathrm{Perf}$.

Note that there is a commutative diagram of fibered categories

$$\begin{tikzcd}
\mathcal{SHT}_{\mathcal{G},\mu/(X,\mathcal{M}_X)^?} \arrow[r, "p_1"] \arrow[dr, "p^\sharp((X,\mathcal{M}_X)^?)"'] & \mathcal{SHT}_{\mathcal{G},\mu} \arrow[r, "p_{\mathrm{Sht}}"] \arrow[dr, "p^\sharp"] & \mathrm{Perf} \\
 & (X, \mathcal{M}_X)^? \arrow[r, "p_2"] & \mathrm{Untilt.} \arrow[u, "\flat"']
\end{tikzcd} \tag{2.9}$$

In the diagram above, $p^\sharp$ (resp. p_2) is a functor sending $(S^\sharp;(\mathscr{P},\phi_{\mathscr{P}}))$ (resp. $(S^\sharp,\mathcal{M}_{S^\sharp},f)$) to $S^\sharp$ with morphisms also projected to the morphisms of the first factor. The fibered category $p^\sharp((X,\mathcal{M}_X)^?) : \mathcal{SHT}_{\mathcal{G},\mu/(X,\mathcal{M}_X)^?} \to (X,\mathcal{M}_X)^?$ is defined by a functor $\mathrm{Sht}_{\mathcal{G},\mu/(X,\mathcal{M}_X)^?}$ sending $(S^\sharp,\mathcal{M}_{S^\sharp},f)$ to $\mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp)$. The functor p_1 is the natural projection. In conclusion, this is the situation to which Appendix A.2 applies. When $? = \Diamond/$, by Definition 2.30, $(X,\mathcal{M}_X)^{\Diamond/}$ also satisfies the diagram above because it is true for both small and big diamonds.

Definition 2.33. *Suppose that $(X,\mathcal{M}_X)$ is in one of the Cases 1-3 in §2.1.3. Define the groupoid of **log $\mathcal{G}$-shtukas bounded by** μ on $(X,\mathcal{M}_X)^?$ as (see Definition A.3)*

$$\mathrm{Sht}^?_{\mathcal{G},\mu}(X,\mathcal{M}_X) := \underset{(S^\sharp,\mathcal{M}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^?)^{\mathrm{op}}}{2\text{-lim}} \mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp),$$

where $? = \Diamond$ or $\diamond$.

Similarly, we define the groupoid of p-adic $\mathcal{G}$-shtukas on $(X,\mathcal{M}_X)^?$ as

$$\mathrm{Sht}^?_{\mathcal{G}}(X,\mathcal{M}_X) := \underset{(S^\sharp,\mathcal{M}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^?)^{\mathrm{op}}}{2\text{-lim}} \mathrm{Sht}_{\mathcal{G}}(S^\sharp).$$

The following statement follows immediately from Lemma A.4.

Proposition 2.34. *With the definitions above, the following two groupoids are canonically* isomorphic*:*

(1) The groupoid $\mathrm{Sht}^?_{\mathcal{G},\mu}(X,\mathcal{M}_X)$;

(2) The groupoid of 1-morphisms between v-stacks on Perf

$$\mathscr{P}^{\log ?} : (X,\mathcal{M}_X)^? \longrightarrow \mathrm{Sht}_{\mathcal{G},\mu}$$

that are sections of $p^\sharp((X,\mathcal{M}_X)^?)$.

We call an object in either of the two groupoids a *log shtuka*.

Definition 2.35. *In general, we can also define $\mathrm{Sht}_{\mathcal{G},\mu}(Y)$ for any v-stack Y mapping to* Untilt *satisfying the diagram (2.9) as the groupoid of 1-morphisms $Y \to \mathrm{Sht}_{\mathcal{G},\mu}$.*

Denote $\mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X,\mathcal{M}_X) := \mathrm{Sht}_{\mathcal{G},\mu}((X,\mathcal{M}_X)^{\Diamond/})$.

For example, we can define $\mathrm{Sht}_{\mathcal{G},\mu}((X,\mathcal{M}_X)^\diamond \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p)$ as a 2-limit over $(X,\mathcal{M}_X)^\diamond \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p$ as in Definition 2.33 or the groupoid of 1-morphisms from $(X,\mathcal{M}_X)^\diamond \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathbb{Q}_p$ to $\mathrm{Sht}_{\mathcal{G},\mu}$.

The following lemma is immediate from definition:

Lemma 2.36. *In all cases for $(X,\mathcal{M}_X)$ in §2.1.3, (strict) étale descent for log shtukas is effective.*

Proof. Let $X' \to X$ be an étale cover. Denote by $S^{\sharp,\prime} \to S^\sharp$ the pullback of $X' \to X$ via $f : S^\sharp \to X$. Since the descent for the étale cover $S^{\sharp,\prime} \to S^\sharp$ is effective for $\mathrm{Sht}_{\mathcal{G},\mu}$ (see [SW20, Prop. 19.5.3]), we have the desired assertion by taking limits. □

2.2.3.

Lemma 2.37. *Let $(X,\mathcal{M}_X)$ be defined as in §2.1.3. Let $S \in$ Perf. Suppose that $X^{\log\Diamond}(S)$ is nonempty. Let $(S^\sharp,\mathcal{M}_{S^\sharp},\alpha_{S^\sharp},f) \in X^{\log\Diamond}(S)$. Then there is a saturated and strongly fine perfectoid log structure $(\mathcal{M}^{\mathrm{can}}_{S^\sharp},\alpha^{\mathrm{can}}_{S^\sharp})$ that satisfies the following universal property:*

For any object $(S^\sharp,\mathcal{M}'_{S^\sharp},\alpha'_{S^\sharp},f) \in X^{\log\Diamond}$ (for the same $S^\sharp$ but varying saturated and fine perfectoid log structures), there is a uniquely determined morphism between log perfectoid spaces $c_{\mathcal{M}'_{S^\sharp}} : (S^\sharp,\mathcal{M}'_{S^\sharp},\alpha'_{S^\sharp}) \to (S^\sharp,\mathcal{M}^{\mathrm{can}}_{S^\sharp},\alpha^{\mathrm{can}})$ extending the identity morphism of the underlying perfectoid spaces.

Moreover, for any morphism $g : (S_1^\sharp, \mathcal{M}_{S_1^\sharp}, \alpha_{S_1^\sharp}) \to (S_2^\sharp, \mathcal{M}_{S_2^\sharp}, \alpha_{S_2^\sharp})$ *in the fibered category* $(X, \mathcal{M}_X)^\Diamond$ *over* Perf*, there is a unique morphism*

$$g^{\mathrm{can}} : (S_1^\sharp, \mathcal{M}^{\mathrm{can}}_{S_1^\sharp}, \alpha^{\mathrm{can}}_{S_1^\sharp}) \to (S_2^\sharp, \mathcal{M}^{\mathrm{can}}_{S_2^\sharp}, \alpha^{\mathrm{can}}_{S_2^\sharp})$$

such that $g^{\mathrm{can}} \circ c_{\mathcal{M}_{S_1^\sharp}} = c_{\mathcal{M}_{S_2^\sharp}} \circ g$*.*

Similar results hold if we replace "$\log \Diamond$*" with "*$\log \diamond$*".*

Proof. We only show the assertions in Case 2 for "$\log \Diamond$"; other cases are proved in the same way. Let us fix $(S^\sharp, \mathcal{M}_{S^\sharp}, \alpha_{S^\sharp}, f)$.

The morphism $f^\sharp : (\mathcal{M}_{S^\sharp}, \alpha_{S^\sharp}) \to (\mathcal{M}_X, \alpha_X)$ gives a commutative diagram:

$$\begin{tikzcd}
& \mathcal{M}_{S^\sharp} \arrow[rr, "\alpha_{S^\sharp}"] \arrow[dd] & & \mathcal{O}_{S^\sharp_{\text{ét}}} \arrow[dd] \\
f^{-1}\mathcal{M}_X \arrow[ur, "f^\sharp"] \arrow[rr, "\alpha"] \arrow[dd] & & f^{-1}\mathcal{O}_{X_{\text{ét}}} \arrow[ur, "f^\sharp"] \arrow[dd] & \\
& \overline{\mathcal{M}}_{S^\sharp} \arrow[rr, "\overline{\alpha}_{S^\sharp}"] & & \mathcal{O}_{S^\sharp_{\text{ét}}}/\mathcal{O}^\times_{S^\sharp_{\text{ét}}} \\
f^{-1}\overline{\mathcal{M}}_X \arrow[ur, "\overline{f}^\sharp"] \arrow[rr, "\overline{\alpha}_X"] & & f^{-1}(\mathcal{O}_{X_{\text{ét}}}/\mathcal{O}^\times_{X_{\text{ét}}}). \arrow[ur, "\overline{f}^\sharp"] &
\end{tikzcd}$$

Let $\overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp} := (f^{-1}\overline{\mathcal{M}}_X)^{\mathrm{perf}} = (\overline{f^*\mathcal{M}_X})^{\mathrm{perf}}$. As $\overline{\mathcal{M}}_{S^\sharp}$ is uniquely p-divisible, there is a unique homomorphism $\overline{\alpha}^{\mathrm{can}} : \overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp} \to \overline{\mathcal{M}}_{S^\sharp}$. Denote by $\mathcal{M}^{\mathrm{canperf}}_{S^\sharp}$ the pullback of $\mathcal{M}_{S^\sharp}$ via $\overline{\alpha}^{\mathrm{can}}$. That is, $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ fits into the following commutative diagram as an extension of $\overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp}$ by $\mathcal{O}^\times_{S^\sharp_{\text{ét}}}$

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}^\times_{S^\sharp_{\text{ét}}} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{can}}_{S^\sharp} \arrow[r] \arrow[d, "c_{\mathcal{M}_{S^\sharp}}"] & \overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp} \arrow[r] \arrow[d, "\overline{\alpha}^{\mathrm{can}}"] & 0 \\
1 \arrow[r] & \mathcal{O}^\times_{S^\sharp_{\text{ét}}} \arrow[r] & \mathcal{M}_{S^\sharp} \arrow[r] & \overline{\mathcal{M}}_{S^\sharp} \arrow[r] & 0.
\end{tikzcd}$$

Let $\alpha^{\mathrm{can}}_{S^\sharp} := \alpha_{S^\sharp} \circ c_{\mathcal{M}_{S^\sharp}}$. We claim that $(\mathcal{M}^{\mathrm{can}}_{S^\sharp}, \alpha^{\mathrm{can}})$ is the desired pair of log structure.

To check the universal property, it suffices to show that the construction is independent of the choice of $f^\sharp : (\mathcal{M}_{S^\sharp}, \alpha_{S^\sharp}) \to (\mathcal{M}_X, \alpha_X)$ for a fixed $f : S^\sharp \to X$. Given two such morphisms $f_1^\sharp : (\mathcal{M}^1_{S^\sharp}, \alpha^1_{S^\sharp}) \to (\mathcal{M}_X, \alpha_X)$ and $f_2^\sharp : (\mathcal{M}^2_{S^\sharp}, \alpha^2_{S^\sharp}) \to (\mathcal{M}_X, \alpha_X)$ that are compatible with a fixed $f : S^\sharp \to X$, we show the definition of $c_{S^\sharp}$ is independent of this choice. Indeed, there is a commutative diagram

(2.10)
$$\begin{tikzcd}
& \overline{\mathcal{M}}^1_{S^\sharp} \arrow[dr, "\overline{\alpha}^1_{S^\sharp}"] & \\
f^{-1}\overline{\mathcal{M}}_X \arrow[ur, "\overline{f}^{1,\sharp}"] \arrow[dr, "\overline{f}^{2,\sharp}"'] & & \mathcal{O}_{S^\sharp_{\text{ét}}}/\mathcal{O}^\times_{S^\sharp_{\text{ét}}} \\
& \overline{\mathcal{M}}^2_{S^\sharp}. \arrow[ur, "\overline{\alpha}^2_{S^\sharp}"'] &
\end{tikzcd}$$

The map $\overline{f}^{i,\sharp}$ uniquely determines a homomorphism $\overline{\alpha}^{\mathrm{can},i} : \overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp} \to \overline{\mathcal{M}}^{i}_{S^\sharp}$ for $i = 1, 2$ and $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ is constructed by pulling back $\mathcal{M}^{i}_{S^\sharp} \to \overline{\mathcal{M}}^{i}_{S^\sharp}$ via $\overline{\alpha}^{\mathrm{can},i}$. We define $(\mathcal{M}^3_{S^\sharp}, \alpha^3_{S^\sharp})$ as $((\mathcal{M}^1_{S^\sharp}, \alpha^1_{S^\sharp}) \oplus (\mathcal{M}^2_{S^\sharp}, \alpha^2_{S^\sharp}))/\mathcal{O}^{\times}_{S^\sharp}$, which is saturated by [Ogu18, I. Prop. 1.3.4] and [Ogu18, I. Prop. 1.3.5]. Moreover, $\overline{\mathcal{M}}^3_{S^\sharp} \cong \overline{\mathcal{M}}^1_{S^\sharp} \oplus \overline{\mathcal{M}}^2_{S^\sharp}$ is uniquely p-divisible. We have the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{M}^1_{S^\sharp} \arrow[dr, "\alpha^1_{S^\sharp}"] & \\
\mathcal{M}^{\mathrm{can}}_{S^\sharp} \arrow[ur, "c_{\mathcal{M}^1_{S^\sharp}}"] \arrow[r, "c_{\mathcal{M}^3_{S^\sharp}}"] \arrow[dr, "c_{\mathcal{M}^2_{S^\sharp}}"'] & \mathcal{M}^3_{S^\sharp} \arrow[u, "p_1"] \arrow[d, "p_2"] & \mathcal{O}_{S^\sharp_{\text{ét}}} \\
 & \mathcal{M}^2_{S^\sharp}, \arrow[ur, "\alpha^2_{S^\sharp}"] &
\end{tikzcd} \tag{2.11}$$

as desired. The functoriality in the third paragraph follows from the construction of $(\mathcal{M}^{\mathrm{can}}_{S^\sharp}, \alpha^{\mathrm{can}}_{S^\sharp})$, $c_{\mathcal{M}_{S^\sharp}}$ and the unique p-divisibility of $\overline{\mathcal{M}}_{S^\sharp_1}$ and $\overline{\mathcal{M}}_{S^\sharp_2}$.

The resulting $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ is saturated and strongly fine perfectoid. Indeed, since $(X, \mathcal{M}_X)$ is fs, by [DLLZ23a, Prop. 2.3.13] and [Ogu18, I. 1.3.6], $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ admits saturated, uniquely p-divisible, and p-finitely generated charts. By [Ogu18, I. Prop. 1.3.4, Prop. 1.3.5], $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ is saturated; it suffices to prove perfectoidness. This follows again from the construction of $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ as a pullback of $\mathcal{M}_{S^\sharp}$ via $\overline{\alpha}^{\mathrm{can}}$ and the perfectoidness of $\mathcal{M}_{S^\sharp}$. ☐

Proposition 2.38. *Denote by* $\mathfrak{f} : X^{\log \Diamond} \to X^{\Diamond}$ *the natural projection. With the conventions as in Lemma 2.37, the object* $(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, \alpha^{\mathrm{can}}, f)$ *is the final object of the fiber of* $\mathfrak{f}$ *at* $(S^\sharp, f)$ *(if the fiber is non-empty). Then a log shtuka*

$$\mathscr{P}^{\log \Diamond} : X^{\log \Diamond} \longrightarrow \mathrm{Sht}_{\mathcal{G},\mu}$$

is uniquely determined by its restriction to the objects $(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, \alpha^{\mathrm{can}})$ *up to isomorphisms. The same is true for "*$\diamond$*".*

Proof. This follows from Lemma 2.37 and Lemma A.6. ☐

Corollary 2.39. *If X is equipped with the trivial log structure* $\mathcal{M}_X = \mathcal{O}^*_{X_{\acute{e}t}}$*, then a log shtuka*

$$\mathscr{P}^{\log \Diamond} : X^{\log \Diamond} \longrightarrow \mathrm{Sht}_{\mathcal{G},\mu}$$

determines and is determined by its restriction to the objects $(S^\sharp, \mathcal{O}^*_{S^\sharp})$*, i.e., a nonlog shtuka: The projection* $X^{\log \Diamond} \to X^{\Diamond}$ *admits a canonical section, and therefore determines a nonlog shtuka. The same is true for "*$\diamond$*" and "*$\Diamond/$*".*

Proof. This follows from the same argument as that of Proposition 2.38. Note that, in this case, the canonical log structures are all trivial. The case of "$\Diamond/$" follows from Definition 2.30 as the corollary is true for both small and big diamonds. ☐

Lemma 2.40. *Continuing with the conventions in Proposition 2.38 with X satisfying (SF), we consider the small diamond* $\mathfrak{f} : X^{\log\diamond} \to X^{\diamond}$*. The map* $f : (S^\sharp := \mathrm{Spa}(R^\sharp, R^{\sharp,+}), \mathcal{M}^{\mathrm{can}}_{S^\sharp}, \alpha^{\mathrm{can}}) \to (X, \mathcal{M}_X)$ *by Definition 2.14 factors through a map* $f^+ : (\mathrm{Spec}\, R^{\sharp,+}, \mathcal{M}_{R^{\sharp,+}}, \alpha_{R^{\sharp,+}}) \to (X, \mathcal{M}_X)$ *such that the pullback log structure* $\widehat{\mathcal{M}}^{\mathrm{ad}}_{R^{\sharp,+}}$ *of* $\mathcal{M}_{R^{\sharp,+}}$ *to* $\mathrm{Spa}(R^{\sharp,+}, R^{\sharp,+})$ *via* $\mathrm{Spa}(R^{\sharp,+}, R^{\sharp,+}) \to \mathrm{Spec}\, R^{\sharp,+}$ *induces the log structure* $\mathcal{M}^{\mathrm{can}}_{S^\sharp}$ *on* $S^\sharp$ *via the pullback through* $S^\sharp \to \mathrm{Spa}(R^{\sharp,+}, R^{\sharp,+})$*. Moreover, f étale locally admits a chart of the form* $\mathsf{P} \hookrightarrow \mathsf{P}[\frac{1}{p}]$.

Proof. The proof of the first assertion follows from the same argument as Lemma 2.37 after the following changes: Here X is a scheme; we replace $\overline{\mathcal{M}}^{\mathrm{canperf}}_{S^\sharp}$ with the log structure induced by

$\overline{\mathcal{M}}_{R^{\sharp,+}} := (f^{+,-1}\overline{\mathcal{M}}_X)^{\mathrm{perf}}$ by pulling back via $S^\sharp \to \operatorname{Spec} R^{\sharp,+}$; the map $\overline{\alpha}^{\mathrm{can}}$ still exists by Lemma 2.12, since the chart, by definition, factors through $R^{\sharp,+}$. The rest of the proof remains unchanged.

For the second assertion, assume that X is affine and admits a global fs sharp chart P. By construction, there is a map $\theta : \mathsf{P} \to \mathcal{M}_{R^{\sharp,+}} \to R^{\sharp,+}$. Since $\mathcal{M}_{R^{\sharp,+}}$ is integral and perfectoid, by [KY25, Rmk. 2.25 and Rmk. 2.26], the induced log structure $\mathcal{M}_{R^{\sharp,+}/p}$ on the mod-p log ring is uniquely p-divisible, and $\theta/p : \mathsf{P} \to \mathcal{M}_{R^{\sharp,+}/p} \to R^{\sharp,+}/p$ can be extended to $\widetilde{\theta/p} : \mathsf{P}[\frac{1}{p}] \to \mathcal{M}_{R^{\sharp,+}/p} \to R^{\sharp,+}/p$. This, in turn, induces a map

$$\xi \circ [\widetilde{\theta/p}^{\flat}] : \mathsf{P}[\frac{1}{p}] \to W(R^+) \to R^{\sharp,+},$$

which induces the log structure $\mathcal{M}_{R^{\sharp,+}}$. The rest of the assertion follows from the construction in the last paragraph. ☐

2.3. **Equivalence of categories on generic fiber.** We discuss a generalization of [PR24, §2.5] in log geometry.

2.3.1. To formulate the equivalence of categories, we need a notion of *(pro-)p-Kummer étale local systems* (see [IKY26])[2].

Let $(X, \mathcal{M}_X)$ be a locally Noetherian fs log adic space. Denote by $\mathrm{Loc}(X_{\text{két}})$ the category of Kummer étale local systems on X with values in finite sets. If we fix a finite set or finite abelian group Λ, we denote by $\Lambda\text{-}\mathrm{Loc}(X_{\text{két}})$ the category of Kummer étale local systems on X with values in Λ. Let ζ be a log geometric point on X defined as in [DLLZ23a, 4.4.2-4.4.3]. Let $\pi_1^{\text{két}}(X, \zeta)$ be the Kummer étale fundamental group at ζ. The following result is a logarithmic analogue of the classical theory that étale local systems correspond to finite étale covers.

Proposition 2.41 ([DLLZ23a, Thm. 4.4.15-Cor. 4.4.18])**.** *Let $(X, \mathcal{M}_X)$ be a connected locally Noetherian fs log adic space. There is a natural equivalence of categories:*

$$X_{\mathit{fk\acute{e}t}} \cong \mathrm{Loc}(X_{\mathit{k\acute{e}t}}) \xrightarrow{\sim} \pi_1^{\mathit{k\acute{e}t}}(X, \zeta)\text{-Fsets}.$$

Let Λ be a finite discrete abelian group. Then the equivalence above restricts to an equivalence

$$\Lambda\text{-}\mathrm{Loc}(X_{\mathit{k\acute{e}t}}) \xrightarrow{\sim} \mathrm{Rep}_\Lambda(\pi_1^{\mathit{k\acute{e}t}}(X, \zeta)).$$

The category $\pi_1^{\mathit{k\acute{e}t}}(X, \zeta)$-Fsets (resp. $\mathrm{Rep}_\Lambda(\pi_1^{\mathit{k\acute{e}t}}(X, \zeta))$) consists of finite discrete sets with continuous $\pi_1^{\mathit{k\acute{e}t}}(X, \zeta)$-actions (resp. continuous $\pi_1^{\mathit{k\acute{e}t}}(X, \zeta)$-representations of Λ).

Proof. Note that the first statement is exactly *loc. cit.*. For any local system $L \in \Lambda\text{-}\mathrm{Loc}(X_{\text{két}})$, the corresponding finite Kummer étale cover Y is Kummer étale locally represented by X^Λ, and the corresponding $\pi_1^{\text{két}}(X, \zeta)$-finite set is given by $\mathrm{Hom}_X(\zeta, Y)$. So the fundamental group $\pi_1^{\text{két}}(X, \zeta)$ acts through $\mathrm{Aut}(\Lambda)$. ☐

The log geometric point ζ is constructed from a geometric point $\xi = \operatorname{Spec} l$ of X. By [DLLZ23a, Cor. 4.4.22], we have that $\pi_1^{\text{két}}(\xi, \zeta) \cong \mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \widehat{\mathbb{Z}}'(1)(l))$, where $\widehat{\mathbb{Z}}'(1)(l) = \varprojlim_n \mu_n(l)$, in which $\mu_n(l)$ is the group of n-th roots of unity in l; the limit runs over all n invertible in l.

Now, assume that X is defined over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Then, in this case, we have $\widehat{\mathbb{Z}}'(1)(l) = \widehat{\mathbb{Z}}(1)$.

Definition 2.42. *Let L be a Kummer étale local system on X with values in Λ. For any log geometric point ζ constructed from a geometric point ξ, L corresponds to a representation $\rho_L \in \mathrm{Rep}_\Lambda(\pi_1^{\mathit{k\acute{e}t}}(X, \zeta))$, and the action of $\pi_1^{\mathit{k\acute{e}t}}(\xi, \zeta)$ on L is given by composing this representation with the natural map $p_\xi^X : \pi_1^{\mathit{k\acute{e}t}}(\xi, \zeta) \to \pi_1^{\mathit{k\acute{e}t}}(X, \zeta)$. Set $\overline{M} := \overline{\mathcal{M}}_{X,\xi}$. In this setting, we say that L*

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[2] We are grateful to Inoue and Koshikawa for suggesting the name "(pro-)p-Kummer"; a similar one "p-primary Kummer" was also suggested to one of us (PW) during his visit to ZJU. In an update [IKY26] to [KY25], Inoue, Koshikawa, and Yao have discussed various equivalent definitions for this notion in more detail. Therefore, the definition used here can be viewed as an adaptation of their work to the context of [DLLZ23a].

*is p-***Kummer at** *ξ if $\rho \circ p_\xi^X$ is trivial when restricted to* $\mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \widehat{\mathbb{Z}}^p(1))$*. We say that L is p-***Kummer** *if it is p-Kummer at every geometric point.*

Next, we consider a pro-Kummer-étale $\widehat{\mathbb{Z}}_p$-local system $\widehat{L}$ on X. By [DLLZ23a, Lem. 6.3.3], $\widehat{L} = \varprojlim_n L_n$, where L_n is a $\mathbb{Z}/p^n\mathbb{Z}$-local system on $X_{\text{két}}$ such that $L_{n+1}/p^n L_{n+1} \cong L_n$. Then, at a log geometric point ζ over a geometric point ξ, the Kummer étale fundamental group $\pi_1^{\text{két}}(X, \zeta)$ has a compatible action on $\{L_{n,\zeta}\}_n$. We obtain a continuous $\mathbb{Z}_p$-representation $\rho_{\widehat{L}} := \varprojlim_n \rho_{L_n}$.

Definition 2.43. *Let $\widehat{L}$ be a pro-Kummer étale $\widehat{\mathbb{Z}}_p$-local system on X. In the setting of Definition 2.42, we say that $\widehat{L}$ is* **pro-p-Kummer at** *ξ if $\rho_{\widehat{L}} \circ p_\xi^X$ is trivial when restricted to* $\mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \widehat{\mathbb{Z}}^p(1))$*. We say that $\widehat{L}$ is* **pro-p-Kummer** *if it is pro-p-Kummer at every geometric point. Denote by* $\widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}_p(X_{\textit{prokét}})$ *the category of pro-p-Kummer étale $\widehat{\mathbb{Z}}_p$-local systems on X.*

Remark 2.44. *Suppose that $\widehat{L}$ is torsion free. Note that the image of* $\mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \widehat{\mathbb{Z}}^p(1))$ *in* $\mathrm{Aut}(\widehat{L}_\zeta)$ *is always finite because a continuous map from the first term (which is a formed by copies of $\widehat{\mathbb{Z}}^p$) to* $\mathrm{Aut}(\widehat{L}_\zeta) \cong \mathrm{GL}_N(\mathbb{Z}_p)$ *always has a finite image. If $\widehat{L}$ has unipotent boundary monodromy (i.e., the image of $\rho_{\widehat{L}} \circ p_\xi^X$ is contained in some $U(\mathbb{Z}_p) \subset GL_N(\mathbb{Z}_p)$ for a unipotent subgroup $U \subset \mathrm{GL}_{N,\mathbb{Z}_p}$), it is always pro-p-Kummer.*

Finally, let $\mathcal{G}$ be the quasi-parahoric $\mathbb{Z}_p$-model of a connected linear algebraic group G over $\mathbb{Q}_p$ in the sense of Definition 1.18.

Definition 2.45. *We say a pro-Kummer étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor $\mathcal{E}$ is* **pro-p-Kummer** *if, for any continuous torsion-free $\mathcal{G}(\mathbb{Z}_p)$-representation V, the corresponding pro-Kummer étale local system $\underline{V} := \mathcal{E} \times^{\mathcal{G}(\mathbb{Z}_p)} V$ is pro-p-Kummer. Denote by* $\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-}\mathrm{Loc}_p(X_{\textit{prokét}})$ *the category of pro-p-Kummer étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors on X.*

2.3.2. Assume that $(X, \mathcal{M}_X)$ is either an fs log scheme locally of finite type over $\mathbb{Q}_p$, or a locally Noetherian fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let G, μ, E, and $\mathcal{G}$ be as in §2.2.

Proposition 2.46. *In the case of locally Noetherian fs log adic spaces over* $\mathrm{Spa}\,\mathbb{Q}_p$*, we have a natural bi-exact tensor equivalence of categories*

$$\widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}_p(X_{\textit{prokét}}) \cong \underset{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f_{S^\sharp}) \in ((X, \mathcal{M}_X)^\diamond)^{\mathrm{op}}}{2\text{-}\lim} \widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}(S^\sharp_{\textit{proét}}). \tag{2.12}$$

By the Tannakian formalism, we have a natural equivalence of categories

$$\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-}\mathrm{Loc}_p(X_{\textit{prokét}}) \cong \underset{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f_{S^\sharp}) \in ((X, \mathcal{M}_X)^\diamond)^{\mathrm{op}}}{2\text{-}\lim} \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-}\mathrm{Loc}(S^\sharp_{\textit{proét}}). \tag{2.13}$$

In the case of locally of finite type schemes, replacing the categories on the left by $\widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}_p(X^{\mathrm{ad}}_{\textit{prokét}})$ *and* $\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-}\mathrm{Loc}_p(X^{\mathrm{ad}}_{\textit{prokét}})$*, we have the same results.*

Lemma 2.47. *Let $(U, \mathcal{M}_U)$ be an affinoid Tate fs log adic space over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ *that admits a global sharp fs chart* P*.*

1. *There is an inverse limit $\varprojlim_i (U_i, \mathcal{M}_i)$ of affinoid finite Kummer étale covers $(U_i, \mathcal{M}_i)$ of $(U, \mathcal{M}_U)$ that is associated with an affinoid perfectoid log adic space $(\widehat{U}_\infty, \widehat{\mathcal{M}}_\infty)$, with $\overline{\widehat{\mathcal{M}}}_\infty$ being uniquely divisible. Moreover, $\widehat{\mathcal{M}}_\infty$ is perfectoid.*
2. *Let $\mathsf{P}[1/p] := \varinjlim_n \frac{1}{p^n}\mathsf{P}$. One can define another perfectoid cover $(\widehat{U}_{p^{-\infty}}, \widehat{\mathcal{M}}_{p^{-\infty}})$ in $U_{\textit{prokét}}$. In fact, there is an inverse limit $U_{p^{-\infty}} := \varprojlim_J \mathrm{Spa}(R_j, R_j^+)$ of affinoid finite étale covers of $U \times_{U\langle \mathsf{P}\rangle} U\langle \mathsf{P}[1/p]\rangle$, which is associated with a log perfectoid space $\widehat{U}_{p^{-\infty}}$ equipped with the log structure $\widehat{\mathcal{M}}_{p^{-\infty}}$ induced by $\mathsf{P}[1/p]$.*

Proof. We prove Part 1. This is essentially [DLLZ23a, Prop. 5.3.12] and [Sch26, Lem. 15.3]. The argument in [DLLZ23a, Prop. 5.3.12] works for both two parts; let us explain more.

In fact, by [Sch13, Prop. 4.8] and [Sch26, Lem. 15.3], for any $U = \mathrm{Spa}(R, R^+)$ as in the Lemma, there is an affinoid perfectoid object $\varprojlim_I U_i = \mathrm{Spa}(R_i, R_i^+)$ in $U_{\text{proét}}$ where R_i are finite étale algebras of R, and $\varprojlim_I U_i \sim \widehat{U}$, that is, is associated with a perfectoid space $\widehat{U}$. Then, by the argument in [DLLZ23a, Prop. 5.3.12], the inverse limits $\varprojlim_{(i,n)\in I\times\mathbb{N}} U_i\langle \frac{1}{n}\mathsf{P}\rangle$ and $\varprojlim_{(i,n)\in I\times\mathbb{N}} U_i\langle \frac{1}{p^n}\mathsf{P}\rangle$ are associated with affinoid log perfectoid spaces $\widetilde{U}_\infty$ and $\widetilde{U}_{p^{-\infty}}$. Since $U \to U\langle \mathsf{P}\rangle$ is a strict closed immersion, the pullback of $\widetilde{U}_\infty \to U\langle \mathsf{P}\rangle$ (resp. $\widetilde{U}_{p^{-\infty}} \to U\langle \mathsf{P}\rangle$) via $U \to U\langle \mathsf{P}\rangle$ is an affinoid perfectoid log adic space $\widehat{U}_\infty$ (resp. $\widehat{U}_{p^{-\infty}}$) in $U_{\text{prokét}}$. The log structures are as desired by construction. ☐

Lemma 2.48 (cf. [IKY26, Prop. 5])**.** *Let $(U, \mathcal{M}_U)$ be a Noetherian affinoid fs log adic space over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ *that admits a global sharp fs chart* P*.*

Let $\widehat{L} \in \widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}(U_{\text{prokét}})$. Then $\widehat{L}$ is in $\widehat{\mathbb{Z}}_p\text{-}\mathrm{Loc}_p(U_{\text{prokét}})$ if and only if $\widehat{L}$ splits over a strict pro-finite étale cover $\widehat{V}_{p^{-\infty}}$ of $\widehat{U}_{p^{-\infty}}$ in the notation of Lemma 2.47(2), such that $\widehat{V}_{p^{-\infty}} \to \widehat{U}_{p^{-\infty}}$ is associated with a pro-finite étale $V_{p^{-\infty}} \to U_{p^{-\infty}}$ in $U_{\text{prokét}}$. In fact, in this case, $\widehat{L}$ splits over $\widehat{V}_{p^{-\infty}}$.

Proof. Without loss of generality, assume that U is connected. Since P is fs and sharp, for any geometric point $\xi \to U$ of U, there is a surjective map $\mathsf{P} \to \overline{\mathcal{M}}_{U,\xi}$ between finitely generated, torsion-free and sharp monoids. Then this induces a surjective map $\pi : \mathsf{P}[1/p]^{\mathrm{gp}}/\mathsf{P}^{\mathrm{gp}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi,p^{-\infty}}/\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi}$. Note that $\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi,p^{-\infty}}/\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi} \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi}, \mathbb{Z}_p(1))$ is a quotient of $\pi_1^{\text{két}}(\xi, \zeta) \cong \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi}, \widehat{\mathbb{Z}}(1))$. Furthermore, we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{P}^{\mathrm{gp}}_\infty/\mathsf{P}^{\mathrm{gp}} \arrow[r, two heads, "\pi_\infty"] \arrow[d] & \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi,\infty}/\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi} \arrow[d] \\
\mathsf{P}[1/p]^{\mathrm{gp}}/\mathsf{P}^{\mathrm{gp}} \arrow[r, two heads, "\pi"] & \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi,p^{-\infty}}/\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi}.
\end{tikzcd} \tag{2.14}$$

Take a log geometric point ζ over ξ. By [DLLZ23a, Cor. 4.4.18], the fiber $\widehat{L}_\zeta$ of $\widehat{L}$ at ζ is a continuous representation of $\pi_1^{\text{két}}(U, \zeta)$. There is an exact sequence

$$1 \to \pi_1^{\text{két}}(\xi, \zeta) \to \pi_1^{\text{két}}(U, \zeta) \to \pi_1^{\text{ét}}(U, \xi) \to 1,$$

where $\pi_1^{\text{ét}}(U, \xi)$ corresponds to the inverse limit of all finite étale covers of U via the equivalence of categories (4.4.19) in [DLLZ23a, Cor. 4.4.18]. Furthermore, the quotient of $\pi_1^{\text{két}}(U, \zeta)$ by the subgroup $\mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{U,\xi}, \widehat{\mathbb{Z}}^p(1)) \subset \pi_1^{\text{két}}(\xi, \zeta)$ corresponds to the cover $\xi_{p^{-\infty}} := U(\xi) \times_{U(\xi)\langle \overline{\mathcal{M}}_{U,\xi}\rangle} U(\xi)\langle \overline{\mathcal{M}}_{U,\xi,p^{-\infty}}\rangle$.

Then $\widehat{L}$ splits over $\widehat{V}_{p^{-\infty}}$ if and only if $\widehat{L}|_\xi$ splits over $\xi \times_U V_{p^{-\infty}}$ for any ξ. Since $\xi_{p^{-\infty}} \to \xi$ factors through $\xi \times_U V_{p^{-\infty}}$, we have the "$\Longleftarrow$" direction.

For the other direction, the action of $\pi_1^{\text{két}}(\xi, \zeta)$ factors through the image of π. By (2.14), the action of $\mathsf{P}^{\mathrm{gp}}_\infty/\mathsf{P}^{\mathrm{gp}}_{p^{-\infty}}$ on $\widehat{L}|_{U(\xi)\times_{U(\xi)\langle \mathsf{P}\rangle} U(\xi)\langle \mathsf{P}_\infty\rangle}$ is trivial by the pro-p-Kummer étale condition at (ξ, ζ). Thus, $\widehat{L}|_{U(\xi)}$ becomes trivial over $U(\xi) \times_{U(\xi)\langle \mathsf{P}\rangle} U(\xi)\langle \mathsf{P}[1/p]\rangle$ by [DLLZ23a, Prop. 4.4.9]. By writing $\widehat{L}$ as an inverse limit of $\mathbb{L}_n$, we find a pro-finite étale cover V of U such that $\widehat{L}$ is trivialized over $V \times_{V\langle \mathsf{P}\rangle} V\langle \mathsf{P}[1/p]\rangle$. Hence, the pullback of $\widehat{L}$ to the perfectoid cover $\widehat{U}_{p^{-\infty}}$ is trivialized over a pro-finite étale cover of $\widehat{U}_{p^{-\infty}}$.

The last sentence can be seen from the proof. ☐

Proof of Proposition 2.46. We only need to show the first equivalence (2.12) as the bi-exactness is clear in the construction. Without loss of generality, we assume that we are in the case of log adic spaces since $(X^{\mathrm{ad}}, \nu^{\mathrm{ad},*}\mathcal{M}_X)$ is locally Noetherian and fs by construction in the case of log schemes. We also assume the connectedness of X.

Since both sides satisfy étale descent, we assume that X is affinoid and that $\mathcal{M}_X$ admits a global chart P.

The RHS of (2.12) is equivalent to $\underset{(S^\sharp, \mathcal{M}_{S^\sharp}, f_{S^\sharp}) \in ((X,\mathcal{M}_X)^\Diamond)^{\mathrm{op}}}{\text{2-lim}} \widehat{\mathbb{Z}}_p\text{-Loc}(S^\sharp_{\text{proét}})$. Let $\{L_{(S^\sharp, \mathcal{M}_{S^\sharp}, f)}\}$ be an object in this limit. This limit gives an object $L_{p^{-\infty}} := L_{(\widehat{X}_{p^{-\infty}}, \widehat{\mathcal{M}}_{p^{-\infty}})}$ by evaluating on the log perfectoid space with a perfectoid log structure $(\widehat{X}_{p^{-\infty}}, \widehat{\mathcal{M}}_{p^{-\infty}})$ formed by Lemma 2.47(2). After replacing $\widehat{X}_{p^{-\infty}}$ with a pro-finite étale cover of it, we can assume that $L_{p^{-\infty}}$ is trivialized. The perfectoid space $\widehat{X}_{p^{-\infty}}$ is, by construction, associated with a perfectoid object $X_{p^{-\infty}}$ in $X_{\text{prokét}}$. Note that $X_{p^{-\infty}} \to X$ is a pro-Kummer étale cover by construction. Then the Kummer étale fundamental group of $(X, \mathcal{M}_X)$ acts on $X_{p^{-\infty}}$, on its associated perfectoid space, and on the trivial local system $L_{p^{-\infty}}$ (which is encoded in the 2-limit). This induces a pro-Kummer étale $\widehat{\mathbb{Z}}_p$-local system $\widehat{L}$ in $\widehat{\mathbb{Z}}_p\text{-Loc}(X_{\text{prokét}})$. Furthermore, $\widehat{L}$ is in $\widehat{\mathbb{Z}}_p\text{-Loc}_p(X_{\text{prokét}})$ by Lemma 2.48.

Let us show the other direction. By Lemma 2.48, an object $\widehat{L} \in \widehat{\mathbb{Z}}_p\text{-Loc}_p(X_{\text{prokét}})$ is trivialized over a perfectoid object $X_{p^{-\infty}}$ in $X_{\text{prokét}}$. Moreover, $\widehat{L}$ is equipped with an action of $\pi_1^{\text{két}}(X, \zeta)$ that factors through $\mathrm{Aut}(X_{p^{-\infty}}/X)$. By applying [DLLZ23a, Lem. 5.3.8] and taking an inverse limit, the saturated product $X_{p^{-\infty}} \times_X^{sat} (X \times_{X\langle \mathsf{P}\rangle} X\langle \mathsf{P}[1/p]\rangle)$ is pro-finite étale over $X \times_{X\langle \mathsf{P}\rangle} X\langle \mathsf{P}[1/p]\rangle$. Since, for every object $(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f) \in (X, \mathcal{M}_X)^\Diamond$, the morphism $f : S^\sharp \to X$ factors through $X \times_{X\langle \mathsf{P}\rangle} X\langle \mathsf{P}[1/p]\rangle$, the saturated product $S^\sharp_{p^{-\infty}} := X_{p^{-\infty}} \times_X S^\sharp$ is pro-finite étale and surjective over $S^\sharp$. The $\pi_1^{\text{két}}(X, \zeta)$-action on $\widehat{L}$ is pulled back to the action of $\mathrm{Aut}(S^\sharp_{p^{-\infty}}/S^\sharp)$ on $(f^{-1}\widehat{L})(S^\sharp_{p^{-\infty}})$. Hence, $f^{-1}\widehat{L}$ is a pro-étale local system on $S^\sharp$. For any morphism $g : (S^\sharp_1, \mathcal{M}_{S^\sharp_1}, f_1) \to (S^\sharp_2, \mathcal{M}_{S^\sharp_2}, f_2)$, it is easy to check that the pullbacks satisfy the desired functoriality required by a 2-limit. ☐

2.3.3. In this subsection, we show the following theorem:

Theorem 2.49 (cf. [PR24, Prop. 2.5.3])**.** *Let $(X, \mathcal{M}_X)$ be a locally Noetherian fs log adic space over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ *(resp. an fs log scheme of finite type over* $\mathrm{Spec}\,\mathbb{Q}_p$*). With this assumption, there is a natural equivalence between the following two categories:*

1. *The category of log shtukas* $\mathrm{Sht}^\Diamond_{\mathcal{G},\mu}(X, \mathcal{M}_X)$*;*
2. *The category $HT_{\mathcal{G},\mu}(X, \mathcal{M}_X)$ of pairs $(\mathbb{P}, \pi_{HT})$, where $\mathbb{P}$ is a pro-p-Kummer-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor defined over X (resp. X^{ad}) and a Hodge-Tate period map $\pi_{HT} : \mathbb{P} \to \mathcal{F}^\Diamond_{G,\mu^{-1}}$.*

The meaning of the last morphism is the following: The torsor $\mathbb{P}$ can be viewed as a limit over $(X, \mathcal{M}_X)^\Diamond$ of pro-étale $\widehat{\mathbb{Z}}_p$-torsors $\mathcal{E}_{S^\sharp}$ by Proposition 2.46. The last morphism is given by a functorial assignment of $\mathcal{G}(\mathbb{Z}_p)$-equivariant maps between v-sheaves $HT(\mathcal{E}_{S^\sharp}) : \mathcal{E}_{S^\sharp} \to$ $\mathrm{Gr}_{G,\mathrm{Spd}\,E,\mu^{-1}}$ *for objects in $(X, \mathcal{M}_X)^\Diamond$ followed by the Bialynicki-Birula isomorphism (see Lemma 1.12)* $\mathrm{Gr}_{G,\mathrm{Spd}\,E,\mu^{-1}} \to \mathcal{F}^\Diamond_{G,\mu^{-1}}$.

Remark 2.50. *See* [IKY26, Thm. 7] *(and also* [KY25, Thm. 7.36]*) for an analogous theorem in the log prismatic theory. In fact, our proof of Theorem 2.49 is also similar to the one presented there.*

Remark 2.51. *The following statement is included in the case of log adic spaces in Theorem 2.49: Let X be a separated $\mathbb{Z}_p$-scheme that is flat and of finite type. Let $\widehat{X}$ be the p-adic completion and $Y := \widehat{X}^{\mathrm{ad}}_\eta$ be the adic generic fiber. If Y is equipped with an fs log structure $\mathcal{M}$, then applying the theorem above gives an equivalence over $(Y, \mathcal{M})^\Diamond$.*

Proof of Theorem 2.49. By definition, we have $\mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X) =$

$$\underset{(S^\sharp,\mathcal{M}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^\Diamond)^{\mathrm{op}}}{\text{2-lim}} \mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp).$$

By [PR24, Prop. 2.5.2], this expression is equivalent to

$$\underset{(S^\sharp,\mathcal{M}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^\Diamond)^{\mathrm{op}}}{\text{2-lim}} HT_{\mathcal{G},\mu}(S^\sharp).$$

The last 2-limit is equivalent to $\underset{(S^\sharp,\mathcal{M}^{\mathrm{can}}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^\Diamond)^{\mathrm{op}}}{\text{2-lim}} HT_{\mathcal{G},\mu}(S^\sharp)$ by Lemma 2.37 and Lemma A.6. Finally, combining Proposition 2.46 (and the explanation in Theorem 2.49), we have the desired equivalence. ☐

Corollary 2.52. *For an fs log scheme $(X,\mathcal{M}_X)$ over $\mathbb{Q}_p$ in §2.1.3 Case 2, we have an equivalence between $\mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X)$ and $HT^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X)$, the category of pairs $(\mathbb{P},\pi_{HT})$ where $\mathbb{P}$ is an object in*

$$\underset{(S^\sharp,\mathcal{M}_{S^\sharp},f_{S^\sharp})\in((X,\mathcal{M}_X)^\Diamond)^{\mathrm{op}}}{\text{2-lim}} \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}(S^\sharp_{\textit{proét}}),$$

and π_{HT} is as above.

Proof. Write as 2-limits and apply [PR24, Prop. 2.5.2]. ☐

Corollary 2.53 (Canonical extension of shtukas in characteristic zero)**.** *(1) Let E be a p-adic field. Assume that X is a smooth rigid analytic variety over* $\mathrm{Spa}\,E$ *with a normal crossings divisor D such that $U := X\backslash D$ is open and dense in X. Denote by $\mathcal{M}_X$ the log structure induced by $D \hookrightarrow X \overset{j}{\hookleftarrow} U$. Then the restriction functor*

$$\mathrm{Res}^X_U : \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X) \to \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(U)$$

is fully faithful.

(2) With the same assumptions as in Part 1, for any object $\mathscr{P}_U \in \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(U)$, $\mathscr{P}_U$ corresponds to a $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor $\mathbb{P}_U$ and a Hodge-Tate map $\pi_{HT} : \mathbb{P}_U \to \mathcal{F}^{\Diamond}_{G,\mu^{-1}}$. Then $\mathscr{P}_U$ is in the essential image of Res^X_U ***only if*** *$\mathbb{P} := j_*\mathbb{P}_U$ is pro-p-Kummer étale.*

*(3) (Diao-Lan-Liu-Zhu) With the same assumptions as in Parts 1 and 2, we further assume that $\mathbb{P}_U$ is **de Rham** in the sense that, for any $V \in \mathrm{Rep}(\mathcal{G}(\mathbb{Z}_p))$, $\underline{V}_{\mathbb{Q}_p} := (\mathbb{P}_U \times^{\underline{\mathcal{G}(\mathbb{Z}_p)}} V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is de Rham. If $\underline{V}_{\mathbb{Q}_p}$ has **unipotent** geometric monodromy along the boundary in the sense of* [DLLZ23a, Def. 6.3.7]*, then there is a canonical way of associating a Hodge-Tate map $\pi_{HT,X} : \mathbb{P} \to \mathcal{F}^{\Diamond}_{G,\mu^{-1}}$ to $\mathbb{P}$, and therefore defining a log shtuka $\mathscr{P} \in \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X)$ via Theorem 2.49. This association is compatible with the one in* [PR24, Prop. 2.6.3] *after the restriction back to $U^\Diamond$.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X,\mathcal{M}_X) \arrow[r] \arrow[d] & \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(U) \arrow[d] \\ HT_{\mathcal{G},\mu}(X,\mathcal{M}_X) \arrow[r] & HT_{\mathcal{G},\mu}(U). \end{tikzcd} \tag{2.15}$$

This diagram is commutative. The vertical maps are equivalences of categories by Theorem 2.49 and [PR24, Prop. 2.5.3]. Hence, for Part 1, it suffices to show that the bottom arrow is fully faithful.

By rigid Abhyankar's Lemma [DLLZ23a, Prop. 4.2.1] and by [DLLZ23a, Cor. 6.3.4], there is an equivalence

$$\widehat{\mathbb{Z}}_p\text{-Loc}(X_{\text{prokét}}) \cong \widehat{\mathbb{Z}}_p\text{-Loc}(U_{\text{proét}}) \tag{2.16}$$

induced by $\widehat{L} \mapsto \widehat{L}|_{U_{\text{proét}}}$; the quasi-inverse is defined by the pushforward of $\widehat{L}|_{U_{\text{proét}}}$ from U to X. By the Tannakian formalism, there is an equivalence

$$\mathrm{Res}^{\mathcal{G}-\mathrm{Loc}} : \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}(X_{\text{prokét}}) \cong \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}(U_{\text{proét}})$$

induced by restricting the pro-Kummer étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor to U. Since the functor $\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}_p(X_{\text{prokét}}) \to \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}(X_{\text{prokét}})$ is fully faithful, there is a fully faithful functor induced by restriction to U:

$$i : \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}_p(X_{\text{prokét}}) \to \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}(U_{\text{proét}}).$$

The rest of Part 1 follows from Lemma 2.54 below.

The quasi-inverse of (2.16) is given by $\widehat{L} \mapsto j_*\widehat{L}$. By the construction above, $j_*\mathbb{P}_U$ in the statement of Part 2 is pro-p-Kummer if and only if $\mathbb{P}_U$ is in the essential image of $\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-Loc}_p(X_{\text{prokét}})$ under the composition $i \circ \mathrm{Res}^{\mathcal{G}-\mathrm{Loc}}$. So Part 2 follows.

Part 3 follows from the main theorems of [DLLZ23b] (see also [RC26, Thm. 4.2.1]). In fact, by the proof of [DLLZ23b, Thm. 3.2.12] written above [DLLZ23b, §3.5], there is a canonical isomorphism

$$\mu^* D_{\mathrm{dR,log}}(\underline{V}) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}} \xrightarrow{\sim} \underline{V} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}},$$

where $\mu : X_{\text{prokét}} \to X_{an}$, $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$ is a peroid sheaf with filtration and log connection extending $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$, and $D_{\mathrm{dR,log}}$ is the arithmetic log Riemann-Hilbert functor (see [DLLZ23b, 2.2 and 3.2.6]). This isomorphism is compatible with filtrations and log connections on both sides.

We then define $\mathbb{M}_{0,\log} := (D_{\mathrm{dR,log}}(\underline{V}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}})^{\nabla^{\log}=0}$, and $\mathbb{M}_{\log} := (\mathrm{Fil}^0(D_{\mathrm{dR,log}}(\underline{V}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}))^{\nabla^{\log}=0}$; they are B^+_{dR}-modules due to log Poincaré's lemma [DLLZ23b, Cor. 2.4.2]. Moreover, the filtrations on them are finite projective modules by [DLLZ23b, Cor. 3.4.22]. Then, using the same argument as in [PR24, §2.6.1], we obtain a Hodge-Tate map $\pi_{HT,X} : \mathbb{P} \to \mathrm{Gr}_{G,\mu^{-1}}$. It is clear that this construction is compatible with and extends the one in *loc. cit.*. □

Lemma 2.54. *With the conventions and assumptions as in Corollary 2.53, every Hodge-Tate map* $\pi_{HT} : \mathbb{P}_U \to \mathcal{F}^{\Diamond}_{G,\mu^{-1}}$ *on* $U^{\Diamond}$ *for a fixed* $\underline{\mathcal{G}(\mathbb{Z}_p)}$*-torsor* $\mathbb{P}_U$ *admits* ***at most one*** *extension to* $\pi_{HT,X} : \mathbb{P} := j_*\mathbb{P}_U \to \mathcal{F}^{\Diamond}_{G,\mu^{-1}}$ *on* $(X, \mathcal{M}_X)^{\Diamond}$.

Proof. Denote by π_1 and π_2 two extensions of π_{HT}. As in the proof of [DLLZ23a, Prop. 4.2.1], let us assume that both $X = \mathrm{Spa}(R, R^+)$ and $U = \mathrm{Spa}(R_0, R_0^+)$ are affinoid. By rigid Abhyankar [DLLZ23a, Prop. 4.2.1] and by Lemma 2.47(1), we can choose an affinoid perfectoid object $\widetilde{U} := \mathrm{Spa}(A, A^+)$ such that $\widetilde{U} \to U$ extends to an affinoid perfectoid object $\widetilde{X} := \mathrm{Spa}(B, B^+) \to X$ in $X_{\text{prokét}}$. We then take the associated perfectoid spaces $\widehat{X} \sim \widetilde{X}$ and $\widehat{U} \sim \widetilde{U}$, and take a strictly totally disconnected cover $\widehat{X}^{str} \to \widehat{X} \to X$ where $\widehat{X}^{str} \to \widehat{X}$ is universally open by [Sch26, Lem. 7.18]. Let $\widehat{U}^{str} := \widehat{X}^{str} \times_{\widehat{X}} \widehat{U}$; this is also a strictly totally disconnected space, since it is an open subspace of $\widehat{X}^{str}$. Note that the log structure on $\widehat{X}^{str}$ is induced by the pullback of the log structure on $\widehat{X}$. Thus, $\widehat{X}^{str}$ with its log structure is an object in $(X, \mathcal{M}_X)^{\Diamond}$.

By assumption, $|U|$ is open dense in $|X|$. In fact, $|\widehat{U}|$ is dense in $|\widehat{X}|$. Indeed, since $|\widehat{X}| \cong \varprojlim_{j\in J} |X_j|$ by [Sch26, Prop. 6.4], it suffices to show that each $U \times_X X_j$ is open dense in X_j. By [Han20, Prop. 2.8], and since the constructions in [Han20] and [DLLZ23a, Prop. 4.2.1] are compatible, the pullback of D through $X_j \to X$ is nowhere-dense. This implies that $U \times_X X_j$ is dense in X_j. Furthermore, we deduce from this fact that $\widehat{U}^{str}$ is also dense in $\widehat{X}^{str}$ since $\widehat{X}^{str} \to \widehat{X}$ is universally open.

If there are two maps $\pi_1(\widehat{X}^{str})$ and $\pi_2(\widehat{X}^{str}) : \mathbb{P}_{\widehat{X}^{str}} \to \mathcal{F}_{G,\mu^{-1},\widehat{X}^{str}}$ extending $\pi_{HT,\widehat{U}^{str}}$, since $|\widehat{U}^{str}|$ is dense in $|\widehat{X}^{str}|$ and $\mathcal{F}_{G,\mu^{-1}}$ is separated over E, we have $\pi_1 = \pi_2$.

Pick an object $(S^\sharp, \mathcal{M}_{S^\sharp}, f) \in (X, \mathcal{M}_X)^{\Diamond}$. Let $\pi_1(S^\sharp)$ and $\pi_2(S^\sharp) : \mathbb{P}_{S^\sharp} \to \mathcal{F}_{G,\mu^{-1},S^\sharp}$ be two Hodge-Tate maps. We now pull them back to the perfectoid space $\widehat{S}^\sharp_{p^{-\infty}}$ associated with the saturated product $S^\sharp \times^{sat}_X \widetilde{X}$, and to $\widehat{S}^\sharp_1 := \widehat{S}^\sharp_{p^{-\infty}} \times_{\widehat{X}} \widehat{X}^{str}$. By functoriality of 2-limits, the two pullbacks $\pi_1(\widehat{S}^\sharp_1)$ and $\pi_2(\widehat{S}^\sharp_1)$ are equal. For the same reason, the v-descent data of the two maps from $\widehat{S}^\sharp_1$ to $S^\sharp$ are canonically identified. Hence, we have $\pi_1(S^\sharp) = \pi_2(S^\sharp)$ by descent. ☐

2.4. **Extending log shtukas.** Let $(X, \mathcal{M}_X)$ be an fs log scheme, where X is a normal scheme that is separated, flat and of finite type over $\mathbb{Z}_p$ or $X = \operatorname{Spec} A$ for an excellent Noetherian normal domain A that is flat over $\mathbb{Z}_p$. Denote by $X_{\mathbb{Q}_p}$ the generic fiber of X. Let $\mathcal{M}_{X_{\mathbb{Q}_p}} = i^*\mathcal{M}_X$ be the log structure induced by the natural inclusion $i : X_{\mathbb{Q}_p} \hookrightarrow X$.

We show that

Theorem 2.55 (cf. [PR24, Thm. 2.7.7])**.** *In the situation above, the restriction functor*

$$\mathrm{Res} := \mathrm{Res}^X_{X_{\mathbb{Q}_p}} : \mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X, \mathcal{M}_X) \longrightarrow \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X_{\mathbb{Q}_p}, \mathcal{M}_{X_{\mathbb{Q}_p}})$$

is fully faithful.

Moreover, the restriction

$$\mathrm{Res} : \mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X, \mathcal{M}_X) \longrightarrow \mathrm{Sht}_{\mathcal{G},\mu}((X, \mathcal{M}_X)^{\diamond} \times_{\operatorname{Spd} \mathbb{Z}_p} \operatorname{Spd} \mathbb{Q}_p)$$

is fully faithful.

2.4.1. Let us introduce some conventions.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two objects in $\mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X, \mathcal{M}_X)$. It follows from the construction in Definition 2.33 and Definition A.3 that $\mathrm{Hom}_{\mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X,\mathcal{M}_X)}(\mathscr{P}_1, \mathscr{P}_2) =$

$$\lim_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f_{S^\sharp}) \in ((X,\mathcal{M}_X)^{\diamond/})^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp)}(\mathscr{P}_{1,(S^\sharp,f)}, \mathscr{P}_{2,(S^\sharp,f)}).$$

The morphisms in $\mathrm{Sht}^{\Diamond}_{\mathcal{G},\mu}(X_{\mathbb{Q}_p}, \mathcal{M}_{X_{\mathbb{Q}_p}})$ are formed in a similar way.

Writing $\overline{\mathscr{P}_i} := \mathscr{P}_i|_{X^{\log\diamond}_{\mathbb{Q}_p}}$ the restriction of $\mathscr{P}_i$ to $X^{\log\Diamond}_{\mathbb{Q}_p}$ for $i = 1, 2$, a morphism $\mathbf{H} \in \mathrm{Ob}\,\mathrm{Hom}(\overline{\mathscr{P}_1}, \overline{\mathscr{P}_2})$ is then represented by $\mathbf{H} = \{\mathbf{H}_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)}\}$, where $\mathbf{H}_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)} \in \mathrm{Hom}_{\mathrm{Sht}_{\mathcal{G},\mu}(S^\sharp)}(\mathscr{P}_{1,(S^\sharp,f)}, \mathscr{P}_{2,(S^\sharp,f)})$ and the collection runs over the objects $(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)$ in $X^{\log\Diamond}_{\mathbb{Q}_p}$ that satisfy the condition in Definition A.3(2).

Similarly, let $\mathcal{H} \in \mathrm{Hom}_{\mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X,\mathcal{M}_X)}(\mathscr{P}_1, \mathscr{P}_2)$ be a morphism. Denote by $\mathcal{H}_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)}$ the morphism between $\mathscr{P}_{1,(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)}$ and $\mathscr{P}_{2,(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)}$ assigned to $\mathcal{H}$.

In what follows, we will frequently use the abbreviations $\mathcal{H}_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)} = \mathcal{H}_{S^\sharp}$ and $\mathbf{H}_{(S^\sharp, \mathcal{M}^{\mathrm{can}}_{S^\sharp}, f)} = \mathbf{H}_{S^\sharp}$ if there is nothing confusing.

Proof of Theorem 2.55. The proof can be adapted from the proofs of [PR24, Prop. 2.7.6] and [PR24, Thm. 2.7.7]. First, note that the argument in *loc. cit.* did not use that the scheme is of finite type after restricting the question locally to a normal domain; moreover, the second claim will follow from the same proof as the first. We assume that we are in the first situation for X.

With the conventions as above, given $\mathbf{H} \in \mathrm{Hom}(\overline{\mathscr{P}_1}, \overline{\mathscr{P}_2})$, we will show that $\mathbf{H}$ extends uniquely to a morphism $\mathcal{H} \in \mathrm{Hom}_{\mathrm{Sht}^{\Diamond/}_{\mathcal{G},\mu}(X,\mathcal{M}_X)}(\mathscr{P}_1, \mathscr{P}_2)$.

Without loss of generality, we assume that $X = \operatorname{Spec} A^+$ is affine and admits a global fs sharp chart P. (Indeed, we can do this by taking an étale neighborhood around each geometric point of X and by taking the log structure on the étale neighborhood the pullback of $\mathcal{M}_X$. After proving this case, one can pass to the Cech nerve of an étale cover.) We write $\widehat{A}^+$ the (p-adic) completion and write $\widehat{A} := \widehat{A}^+[\frac{1}{p}]$. Write $\widehat{X} := \operatorname{Spf} \widehat{A}^+$ and $\widehat{X}^{\mathrm{ad}} := \operatorname{Spa}(\widehat{A}^+, \widehat{A}^+)$.

Let $f : (Y := \operatorname{Spa}(B, B^+), \mathcal{M}_Y) \to (\widehat{X}^{\mathrm{ad}}, \mathcal{M}_{\widehat{X}^{\mathrm{ad}}})$ be an object in $(X, \mathcal{M}_X)^{\diamond}$. Here, we also assume that $Y = \operatorname{Spa}(B, B^+)$ is affinoid perfectoid and assume that $\mathcal{M}_Y = \mathcal{M}^{\mathrm{can}}_Y$.

Firstly, we consider the case where $(B, B^+) = (C, C^+)$ is an algebraically closed, (nonarchimedean, complete) perfectoid field. As in the proof of [PR24, Prop. 2.7.6], if the morphism f is adic, then C is of characteristic zero, but, in this case, we have that f is already an object in $(X_{\mathbb{Q}_p}, \mathcal{M}_{X_{\mathbb{Q}_p}})^{\diamondsuit}$, so we set $\mathcal{H}_{\mathrm{Spa}(C,C^+)} = \mathbf{H}_{\mathrm{Spa}(C,C^+)}$.

If the morphism f is not adic, then C is of characteristic p. In this case, we claim that $f : \mathrm{Spa}(C, C^+) \to \widehat{X}^{\mathrm{ad}}$ factors through an integral perfectoid space X_∞ over $\widehat{X}^{\mathrm{ad}}$. In fact, let $\widehat{X}_0 := \mathrm{Spa}(\widehat{A}, \widehat{A}^+)$. Then $\widehat{X}_0$ is an affinoid Tate adic space equipped with a morphism to the adic generic fiber of $\widehat{X}^{\mathrm{ad}}$ and to $\widehat{X}^{\mathrm{ad}}$, $i : \widehat{X}_0 \to \widehat{X}^{\mathrm{ad}}_\eta \to \widehat{X}^{\mathrm{ad}}$. Let $R_{p-\infty}$ be the inverse limit of finite étale algebras over $\widehat{A}_{p-\infty} := \widehat{A} \otimes_{\widehat{A}\langle \mathsf{P}\rangle} \widehat{A}\langle \mathsf{P}[1/p]\rangle$. Let $R^+_{p-\infty}$ be the integral closure in $R_{p-\infty}$ of $\widehat{A}^+$. By Lemma 2.47(2), the p-adic completion $(\widehat{R}_{p-\infty}, \widehat{R}^+_{p-\infty})$ is a complete affinoid perfectoid Huber pair. Now, since (C, C^+) is an algebraically closed perfectoid field, C^+ is a valuation ring and $R^+_{p-\infty}$ is formed by an inverse limit of normalizations in finite ring extensions, the map $\widehat{A}^+ \to C^+$ factors through $\widehat{A}^+ \to \widehat{R}^+_{p-\infty} \to C^+$. Set $X_\infty := \mathrm{Spa}(\widehat{R}^+_{p-\infty})$. The claim is shown.

By Lemma 2.47(2), X_∞ is equipped with the log structure $\mathsf{P}[1/p]^a$ induced by $\mathsf{P}[1/p]$; this is a saturated and fine perfectoid log structure by construction (see also [KY25, Rmk. 2.26]). So the associated log perfectoid space $X_{\infty,\eta} := \mathrm{Spa}(\widehat{R}_{p-\infty}, \widehat{R}^+_{p-\infty})$ with its natural morphism to $\widehat{X}^{\mathrm{ad}}$ is an object in $X^{\log \diamond}$. By [PR24, Prop. 2.7.6], $\mathbf{H}_{X_{\infty,\eta}}$ extends uniquely to a morphism $\mathcal{H}_{\mathrm{Spd}(\widehat{R}^+_{p-\infty})} : \mathscr{P}1|_{\mathrm{Spd}(\widehat{R}^+_{p-\infty})} \to \mathscr{P}2|_{\mathrm{Spd}(\widehat{R}^+_{p-\infty})}$, and $\mathcal{H}_{\mathrm{Spa}\, C}$ is determined by pulling back $\mathcal{H}_{\mathrm{Spd}(\widehat{R}^+_{p-\infty})}$ to $\mathrm{Spd}\, C$. (To apply the proposition there, one needs that $\widehat{R}^+_{p-\infty} = \widehat{R}^\circ_{p-\infty}$. This follows from the normalization construction.)

Finally, we consider general $Y = \mathrm{Spa}(B, B^+)$. This follows from a v-descent argument. As in the proof of *loc. cit.*, we choose a collection of maps $\{\pi_i : \mathrm{Spa}(C_i, C_i^+) \to \mathrm{Spa}(B, B^+)\}$ covering $|Y|$, where $\mathrm{Spa}(C_i, C_i^+)$ are algebraically closed perfectoid fields. We then form a product $Z := \mathrm{Spa}(\prod_i C_i, \prod_i C_i^+)$ with a v-cover $\pi := \prod_i \pi_i : Z \to Y$. Then Z maps to $\widehat{X}^{\mathrm{ad}}$ via $Z \xrightarrow{\pi} Y \xrightarrow{f} \widehat{X}^{\mathrm{ad}}$, and the log structure on Z is defined by pulling back $\mathcal{M}_Y$. We can construct $\mathcal{H}_Z$: By the proof of [PR24, Prop. 2.7.6], the morphisms on $W(C_i^+)$ for the corresponding BKF modules. We then take the product of these morphisms on $\prod_i W(C_i^+)$ and restrict to get a morphism $\mathcal{H}_Z$ between shtukas.

To show that the morphism $\mathcal{H}_Z$ descends to a unique morphism $\mathcal{H}_Y$, it suffices to check that there is a descent datum $p_1^*\mathcal{H}_Z \cong p_2^*\mathcal{H}_Z$ on $\widetilde{Z} := Z \times_Y Z$. As explained in the proof of *loc. cit.*, it suffices to check this pointwisely. For any point $s : \mathrm{Spa}(C, C^+) \to \widetilde{Z}$, we have $\pi \circ p_1 \circ s = \pi \circ p_2 \circ s$. So the two sides are the equal after post-composing f. It suffices to deal with the case that C is of characteristic p. By the last three paragraphs, we know that $f \circ \pi \circ p_i \circ s$ can also be factored as $x : \mathrm{Spa}\, C \to \mathrm{Spd}\, C^+ \to \mathrm{Spd}\, \widehat{R}^+_{p-\infty} \to X^{\log \diamond}$, and this factorization is *unique* by the first paragraph of the proof of *loc. cit.* (see [PR24, 2.7.2] and [Gle25]). Thus the last two paragraphs gives a unique way of defining $\mathcal{H}_{\mathrm{Spa}\, C, x}$, as desired.

We now check the uniqueness (as there might be a different choice of v-cover) and functoriality of this assignment. Given a morphism $g : (Y, \mathcal{M}_Y^{\mathrm{can}}, f) \to (Y', \mathcal{M}_{Y'}^{\mathrm{can}}, f')$ such that $f' \circ g = f$, we check that, taking products of points $Z \to Y$ and $Z' \to Y'$ and constructing $\mathcal{H}_Y$ and $\mathcal{H}_{Y'}$ using them respectively, we have $g^*\mathcal{H}_{Y'} = \mathcal{H}_Y$. Upon replacing Z with the disjoint union of it with a v-cover of $Y \times_{Y'} Z'$, we assume that there is a morphism $\widetilde{g} : Z \to Z'$ covering g. Then, by the same argument as the last paragraph, the $\mathcal{H}_{\mathrm{Spa}\, C, x}$ at any point of Z constructed by either pulling back from Z or pulling back from $Z \xrightarrow{\widetilde{g}} Z'$ are the same as the one factoring through $\mathrm{Spd}\, \widehat{R}^+_{p-\infty}$. So $\widetilde{g}^*\mathcal{H}_{Z'} = \mathcal{H}_Z$. Next, we consider the fiber product $\widetilde{Z}$ as the last paragraph, and the same argument shows that the morphisms descend and are equal on Y.

For the case of affine excellent normal flat domains, all arguments above work; note that the normalization argument goes through by [GD64, part2, 7.8.3(5)]. ☐

2.4.2. Let us state an immediate corollary of Theorem 2.55.

Corollary 2.56 (cf. [PR24, Rmk. 2.7.9])**.** *Let X be a normal scheme that is flat, separated and of finite type over $\mathbb{Z}_p$. Let $D \subset X$ be a relative Cartier divisor such that $U := X \backslash D$ is open dense in X. Let $\mathcal{M}_X$ be the log structure defined by $D \hookrightarrow X \hookleftarrow U$.*

(1) Suppose that the generic fiber $X_{\mathbb{Q}_p}$ is a smooth variety with the normal crossings divisor $D_{\mathbb{Q}_p}$. Then the restriction functor

$$\mathrm{Res}_U^X : \mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond/}(X, \mathcal{M}_X) \to \mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond/}(U)$$

is fully faithful.

(2) Suppose that $X_{\mathbb{Q}_p}$ is a smooth variety, and that $\widehat{U}_\eta^{\mathrm{ad}} \hookrightarrow \widehat{X}_\eta^{\mathrm{ad}}$ has a normal crossings complement. Then the restriction functor

$$\mathrm{Res}_U^X : \mathrm{Sht}_{\mathcal{G},\mu}^{\diamond}(X, \mathcal{M}_X) \to \mathrm{Sht}_{\mathcal{G},\mu}^{\diamond}(U)$$

is fully faithful.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond/}(X, \mathcal{M}_X) \arrow[r, "\mathrm{Res}_U^X"] \arrow[d] & \mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond/}(U) \arrow[d] \\
\mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond}(X_{\mathbb{Q}_p}, \mathcal{M}_{X_{\mathbb{Q}_p}}) \arrow[r, "\mathrm{Res}_{U_{\mathbb{Q}_p}}^{X_{\mathbb{Q}_p}}"] & \mathrm{Sht}_{\mathcal{G},\mu}^{\Diamond}(U_{\mathbb{Q}_p}).
\end{tikzcd}$$

Then the assertion follows from combining Theorem 2.55, [PR24, Thm. 2.7.7] and Corollary 2.53. Indeed, we have known that the functors represented by the bottom arrow and the vertical arrows are fully faithful.

The second part follows similarly from Remark 2.51, Corollary 2.53 and Theorem 2.55. ☐

2.5. **Restriction to special fiber.** Suppose that $(X, \mathcal{M}_X)$ is in Case 2 of log schemes.

2.5.1. Assume that X is defined over $\mathbb{Z}_p$ or just $\mathbb{F}_p$.

Proposition 2.57. *The projection $F^{\log} : (X_{\mathbb{F}_p}, \mathcal{M}_{X_{\mathbb{F}_p}})^\diamond \to X_{\mathbb{F}_p}^\diamond$ admits a canonical section $i^{\log}$. This section is given by assigning to any $(S^\sharp, f) \in X_{\mathbb{F}_p}^\diamond(S)$ an object in log diamond $(S^\sharp, (f^*\mathcal{M}_{X_{\mathbb{F}_p}})^{perf}, f) \in X_{\mathbb{F}_p}^{\log \diamond}$. The same holds after replacing $\diamond$ with $\Diamond$.*

Proof. When $S^\sharp$ is an affinoid perfectoid space of characteristic p, it is perfect. Then the morphism between sheaves of monoids $f^*\mathcal{M}_{X_{\mathbb{F}_p}} \to \mathcal{O}_{S^\sharp_{\text{ét}}}$ induces a morphism $(f^*\mathcal{M}_{X_{\mathbb{F}_p}})^{\mathrm{perf}} \to \mathcal{O}_{S^\sharp_{\text{ét}}}$. ☐

Corollary 2.58. *Suppose that there is a log $\mathcal{G}$-shtuka bounded by μ,*

$$\mathscr{P} : (X, \mathcal{M}_X)^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

Then the restriction of $\mathscr{P}$ to $(X_{\mathbb{F}_p}, \mathcal{M}_{X_{\mathbb{F}_p}})^\diamond$ gives a morphism

$$\mathscr{P}|_{X_{\mathbb{F}_p}} : (X_{\mathbb{F}_p}, \mathcal{M}_{X_{\mathbb{F}_p}})^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

This morphism is equivalent to a morphism

$$\mathscr{P}|_{X_{\mathbb{F}_p}} : X_{\mathbb{F}_p}^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}.$$

The same holds if we replace $\diamond$ with $\Diamond$.

Moreover, there is a morphism $\mathscr{P}^{\mathrm{red}} : X_{\mathbb{F}_p}^{perf} \to \mathrm{Sht}_{\mathcal{G},\mu}^W$.

Proof. The construction of $\mathcal{M}_{S^\sharp}^{\mathrm{can}}$ in Lemma 2.37 coincides with the perfection of $f^*\mathcal{M}_X$ in Proposition 2.57. By Proposition 2.38, we have a morphism $X_{\mathbb{F}_p}^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}$. Applying the reduction functor in the sense of [Gle25], and noting that $(\mathrm{Sht}_{\mathcal{G},\mu})_{\mathrm{red}}$ is represented by $\mathrm{Sht}_{\mathcal{G},\mu}^W$, $X_{\mathbb{F}_p}^\diamond = (X_{\mathbb{F}_p}^{\mathrm{perf}})^\diamond$ by [SW20, Prop. 18.3.1] and that $((X_{\mathbb{F}_p}^{\mathrm{perf}})^\diamond)_{\mathrm{red}}$ is represented by $X_{\mathbb{F}_p}^{\mathrm{perf}}$ by [Gle25, Prop. 3.16], we get the desired morphism. ☐

For simplicity, we will use charts to refer to log structures in the remaining part of §2.5.

Remark 2.59. *Let K be a finite field extension of $\mathbb{Q}_p$, and let $\varpi \in K$ be a uniformizer. One can consider the absolute log prismatic site $(\mathcal{O}_K, \varpi^{\mathbb{N}})_{\mathbb{\Delta}}$. A log prismatic F-crystal $\mathcal{E}$ on $(\mathcal{O}_K, \varpi^{\mathbb{N}})_{\mathbb{\Delta}}$ is equivalent to a semistable G_K-representation $\mathbb{L}$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}}(G_K)$ by* [Yao23]*. One can read off the monodromy action of the semistable representation $\mathbb{L}[1/p]$ by evaluating the associated log prismatic F-crystal $\mathcal{E}$ on Breuil-Kisin/Hyodo–Kato log prisms. If we evaluate $\mathcal{E}$ at the cover $(\mathcal{O}_K, \varpi^{\mathbb{N}}) \to (\mathcal{O}_C, \varpi^{\mathbb{N}[\frac{1}{p}]})$, one obtains a Breuil-Kisin-Fargues module (see* [Yao23, Ex. 3.1]*). Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$. The objects $(S^\sharp, \mathcal{M}_{S^\sharp}, f) \in (\mathcal{O}_K, \varpi^{\mathbb{N}})^\diamond(S)$ require $S^\sharp$ to be perfectoid and $\mathcal{M}_{S^\sharp}$ to be saturated and fine perfectoid; thus $\overline{\mathcal{M}}_{S^\sharp}$ is uniquely p-divisible (see Lemma 2.10). In particular, under the equivalence of categories between perfect log prisms and perfectoid log rings in* [KY25, Prop. 2.39]*, one only sees the* perfect *objects in $(\mathcal{O}_K, \varpi^{\mathbb{N}})^\diamond$. Therefore, the realization functor from log prismatic F-crystals to log shtukas forgets the monodromy action when we consider the specialization map as in* [PR24, Prop. 2.4.6].

2.5.2. *Log p-divisible groups.* In this subsection, we give an example that might be relevant to Corollary 2.58. Proposition 2.60 will not be used in the remaining of this paper.

Let K be a finite field extension of $\mathbb{Q}_p$, let $\mathcal{O}_K$ be its ring of integers, and let k be its residue field. Let $\varpi \in K$ be a uniformizer, and set $S = (\mathrm{Spec}\, \mathcal{O}_K, \varpi^{\mathbb{N}})$, $S_0 = (\mathrm{Spec}\, k, 0^{\mathbb{N}})$. By the main theorem of [BWZ23], taking the generic fiber $(\cdot)_K$ gives an equivalence of categories between the category $\mathbf{BT}_{S,d}^{\log}$ of dual-representable log p-divisible groups over S and the category $\mathbf{BT}_K^{\mathrm{st}}$ of p-divisible groups over K with semistable reduction. Moreover, taking the Tate module T_p gives an equivalence of categories between $\mathbf{BT}_K^{\mathrm{st}}$ and the category $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st},\{0,1\}}(G_K)$ of semistable G_K-representations with Hodge–Tate weights in $\{0, 1\}$.

Let $\mathcal{H} \in \mathbf{BT}_{S,d}^{\log}$. On the one hand, as in Remark 2.59, one can attach to it a log prismatic F-crystal and then a log shtuka in $(\mathcal{O}_K, \varpi^{\mathbb{N}})^\diamond$; by restricting to the special fiber and by Corollary 2.58, we obtain a usual shtuka over $\mathrm{Spec}\, k$. On the other hand, the special fiber $\mathcal{H}_k$ of $\mathcal{H}$ is a log p-divisible group over S_0, which need not be a classical p-divisible group. Nevertheless, one can still produce a usual shtuka from $\mathcal{H}_k$ that is compatible with the above one, by Lemma 2.37 and the following result:

Proposition 2.60. *Let $\mathcal{H}$ be a dualizable log p-divisible group over* $(\mathrm{Spec}\, k, 0^{\mathbb{N}})$*. Then its pullback to* $(\mathrm{Spec}\, k, 0^{\mathbb{N}[\frac{1}{p}]})$ *becomes a classical p-divisible group.*

Proof. This follows from Kato's classification of log p-divisible groups [Kat23, Thm. 3.1]; cf. [WZ24, Thm. 3.8, Cor. 3.10]. We briefly recall some notation and results from Kato's work on log p-divisible groups, following [WZ24, §3]. Let $T = \mathrm{Spec}\, A$, where A is a Noetherian henselian local ring with the residue characteristic p, and suppose that T admits a global chart $P \to \mathcal{M}_T$ such that the induced map $\mathsf{P} \to \mathcal{M}_{T,\bar{t}}/\mathcal{O}_{T,\bar{t}}^\times$ is an isomorphism at the geometric point $\bar{t}$ of T. Let $(\mathrm{fin}/T)_d$ be the category of finite Kummer flat log group schemes over T defined as in [Kat23, 1.6]; see also [WZ24, Def. 2.3, 2.6] for details. For $\mathcal{G} \in (\mathrm{fin}/T)_d$, there is a unique exact sequence $0 \to \mathcal{G}^\circ \to \mathcal{G} \to \mathcal{G}^{\text{ét}} \to 0$, which restricts to the classical connected-étale sequence over any finite Kummer flat cover where $\mathcal{G}$ becomes a classical p-divisible group. Moreover, $\mathcal{G}^\circ$ and $\mathcal{G}^{\text{ét}}$ are classical finite flat group schemes. From Kato's classification, any $\mathcal{G} \in (\mathrm{fin}/T)_d$ that is p-power torsion determines, and is uniquely determined by, a pair $(\mathcal{G}^{\mathrm{cl}}, \beta)$, where $\mathcal{G}^{\mathrm{cl}}$ is a classical extension of $\mathcal{G}^\circ$ and $\mathcal{G}^{\text{ét}}$, and $\beta \in \mathrm{Hom}(\mathcal{G}^{\text{ét}}(1), \mathcal{G}^\circ) \otimes_{\mathbb{Z}} \mathsf{P}^{\mathrm{gp}}$.

By definition, $\mathcal{H} = \varinjlim \mathcal{H}_n$, where $\mathcal{H}_n = \ker(\mathcal{H} \xrightarrow{p^n} \mathcal{H})$ is an object in $(\mathrm{fin}/S_0)_d$. Therefore, $\mathcal{H}_n$ determines and is determined by $(\mathcal{H}_n^{\mathrm{cl}}, \beta_n)$. Since $\mathrm{Hom}(\mathcal{H}_n^{\text{ét}}(1), \mathcal{H}_n^\circ)$ is killed by p^n, $\beta \in \mathrm{Hom}(\mathcal{H}_n^{\text{ét}}(1), \mathcal{H}_n^\circ) \otimes_{\mathbb{Z}} \mathsf{P}^{\mathrm{gp}}/p^n \mathsf{P}^{\mathrm{gp}}$. In particular, after passing to $S_{0,\infty} = (\mathrm{Spec}\, k, 0^{\mathbb{N}[\frac{1}{p}]})$, β is trivialized and $\mathcal{H}_n$ becomes a classical p-divisible group. Therefore, $\mathcal{H}$ is classical over $(\mathrm{Spec}\, k, 0^{\mathbb{N}[\frac{1}{p}]})$. $\square$

3. LOCAL SYSTEMS AND SHTUKAS ON MIXED SHIMURA VARIETIES

In this section, we work with the theory in characteristic 0.

3.1. **Group-theoretic lemmas.** We collect the results used later regarding various group-theoretic constructions in compactification theory, Bruhat-Tits theory, and their intersection.

3.1.1. Let P be any linear algebraic group. Define $P^c := P/Z(P)_{ac}$, where $Z(P)_{ac}$ is the anti-cuspidal part of the multiplicative connected center $Z(P)^\circ$ of P.

Given a mixed Shimura datum $(P, \mathcal{X})$, let G be the Levi quotient of P. Then the natural projection $P \to G$ induces an embedding $Z(P) \hookrightarrow Z(G)$, and $Z(P)_{ac}$ is sent to $Z(G)_{ac}$ by definition. Note that $P^c \to G^c$ factors through the Levi quotient $G^{\prime,c}$ of P^c, and $G^{\prime,c} \to G^c$ is a homomorphism with central kernel.

Lemma 3.1. *For a mixed Shimura datum $(P, \mathcal{X})$, we have $G^{\prime,c} = G^c$.*

Proof. Recall that $(P, \mathcal{X})$ satisfies [Pin90, 2.1 (viii)]. That is, we require that, for any Levi subgroup $\widetilde{G} \subset P$ lifting G, the adjoint action of the center $Z(\widetilde{G})$ of $\widetilde{G}$ on $\mathrm{Lie}\, W$ factors through a cuspidal quotient; W denotes the unipotent radical of P. Note that the definition of $Z(\widetilde{G})$ is independent of the choice of lifting $\widetilde{G}$. From this condition, we see that

$$Z(G)_{ac} \xrightarrow{\sim} Z(\widetilde{G})_{ac} \subset \mathrm{Cent}_{Z(\widetilde{G})}(\mathrm{Lie}\, W) = Z(P).$$

So the embedding $Z(P)_{ac} \hookrightarrow Z(G)_{ac}$ admits a section by the displayed expression above. We then have $Z(P)_{ac} = Z(G)_{ac}$, and the lemma follows. $\square$

3.1.2. Now let (G, X) be a Shimura datum. Let $\Phi = (Q_\Phi, X_\Phi^+, g_\Phi)$ be a cusp label representative of (G, X) (see [MP19, 2.1.7]). Recall that one associates to Φ a mixed Shimura datum (P_Φ, D_Φ). Recall that $Q_\Phi \subset G$ is an admissible $\mathbb{Q}$-parabolic that contains a normal subgroup P_Φ and is equipped with a $Q_\Phi(\mathbb{R})$-equivariant morphism

$$\tau : X \to \pi_0(X) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\Phi,\mathbb{C}}), \quad x \mapsto ([x], u_x^{Q_\Phi} \circ h_\infty).$$

Moreover, D_Φ is defined as the $P_\Phi(\mathbb{R})U_\Phi(\mathbb{C})$-orbit of a fixed point $([x], u_x^{Q_\Phi} \circ h_\infty)$ such that $x \in X_\Phi^+$, where X_Φ^+ is a connected component of X; it depends only on X_Φ^+, not on the choice of x in X_Φ^+. Denote by W_Φ the unipotent radical of P_Φ and U_Φ the center of W_Φ. Let K be an open compact subgroup of $G(\mathbb{A}_f)$. Define $K_\Phi = P_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}$; define $K_{\Phi,p}$ and K_Φ^p similarly.

Lemma 3.2. *The natural embedding $P_\Phi \to G$ induces a finite map $P_\Phi^c \to G^c$. Let $ZP_\Phi := (Z_G \cdot P_\Phi)^\circ$. Then there is an injective map $ZP_\Phi^c \to G^c$.*

Proof. Since the center of ZP_Φ/ZW_Φ is isogenous to a product of split tori and $\mathbb{R}$-anisotropic tori (cf. [Pin90, Cor. 4.10] and [Wu25, Lem. 1.14]), we know that $Z(P_\Phi)_{ac} \subset Z(ZP_\Phi)_{ac} = Z(ZW_\Phi)_{ac} = Z(G)_{ac}$. So there is a well-defined quotient map $P_\Phi^c = P_\Phi/Z(P_\Phi)_{ac} \to G^c = G/Z(G)_{ac}$. For ZP_Φ, by the last paragraph, we see that the embedding $ZP_\Phi \hookrightarrow G$ induces an embedding $ZP_\Phi^c \hookrightarrow G^c$. $\square$

Lemma 3.3. *Let $\mathcal{G}$ be a quasi-parahoric group scheme of G, and let $\mathcal{G}_\Phi$ be the g_Φ-conjugate of $\mathcal{G}$ (i.e. $\mathcal{G}_\Phi(\breve{\mathbb{Z}}_p) = g_\Phi \mathcal{G}(\breve{\mathbb{Z}}_p) g_\Phi^{-1}$). Let $\mathcal{Q}_\Phi$ (resp. $\mathcal{W}_\Phi$) be the closure of Q_Φ (resp. W_Φ) in $\mathcal{G}_\Phi$, and let $\mathcal{P}_\Phi$ be the smoothing of the closure of P_Φ in $\mathcal{G}_\Phi$.*

(1) When $\mathcal{G}$ is stabilizer quasi-parahoric (resp. quasi-parahoric, parahoric), then $\mathcal{Q}_\Phi$ is stabilizer quasi-parahoric (resp. quasi-parahoric, parahoric) in the sense of 1.18. Moreover, when $\mathcal{G}$ is parahoric, then $\mathcal{Q}_\Phi \hookrightarrow \mathcal{G}_\Phi$ is a parabolic embedding in the sense of 1.19.

(2) When $\mathcal{G}$ is stabilizer quasi-parahoric (resp. quasi-parahoric, parahoric), then $\mathcal{P}_\Phi$ is stabilizer quasi-parahoric (resp. quasi-parahoric, quasi-parahoric). Moreover, $(\mathcal{P}_\Phi, \mu_\Phi)$ comes from boundary in the sense of 1.21.

Let $\mathcal{G}$ be quasi-parahoric. Let $\mathcal{U}_\Phi$ be the closure of U_Φ in $\mathcal{W}_\Phi$, and let $\mathcal{V}_\Phi$ be the quotient $\mathcal{W}_\Phi/\mathcal{U}_\Phi$. Then $\mathcal{U}_\Phi$ and $\mathcal{V}_\Phi$ are affine smooth group schemes with connected fibers, and we have the decomposition $\mathcal{W}_\Phi = \mathcal{U}_\Phi \rtimes \mathcal{V}_\Phi$.

Proof. This is essentially [Mao25a, Prop. 2.67]. We start from $\mathcal{G}_\Phi$ and omit the index Φ for simplicity (we won't mention the initial $\mathcal{G}$). First of all, assume $K_p := K_{\Phi,p}$ and $Q := Q_\Phi$ are in good position, i.e. the associated (generic) point $x \in B_{\text{red}}(G, \mathbb{Q}_p)$ of the facet $\mathcal{F} \subset B_{\text{red}}(G, \mathbb{Q}_p)$ ($\mathcal{G}^\circ = \mathcal{G}_\mathcal{F}$) is contained in an apartment $A_{\text{red}}(G, T)$, where T is a maximal $\breve{\mathbb{Q}}_p$-split torus of G defined over $\mathbb{Q}_p$ such that $T \subset L \subset Q$. By the arguments below Definition 1.18, the arguments above Definition 1.21, and by [Mao25a, Prop. 2.67] with the help of Lemma 3.4, we see that both statements (1) and (2) are true. In this case, $\mathcal{L}$ (resp. $\mathcal{G}_h$) is the smoothening of the closure of L (resp. G_h) in $\mathcal{G}$.

In general, we first show (1): apply arguments in [Mao25a, Rmk. 2.45]: we can always find an element $g \in G(\mathbb{Q}_p)$ such that $K' := g^{-1}Kg$ and Q are in good position, $K = gK'g^{-1}$. We factor $g = qnk$, $q \in Q(\mathbb{Q}_p)$, $n \in N(\mathbb{Q}_p)$, $k \in P_x^0$, where N is the normalizer of T over $\mathbb{Q}_p$, and $P_x^0 = \mathcal{G}_x^\circ(\mathbb{Z}_p) = (K')^\circ$. The action k fixes x, n moves x to $nx \in A_{\text{red}}(G, T)$, and q shifts the apartment $A_{\text{red}}(G, T)$ in $B_{\text{red}}(G, \mathbb{Q}_p)$. Let $K_1 = nkK'(nk)^{-1}$, then K_1 and Q are in good position, $Q(\mathbb{Q}_p) \cap K_1 = (W(\mathbb{Q}_p) \cap K_1) \rtimes (L(\mathbb{Q}_p) \cap K_1)$ by the first paragraph. Since W is normal in Q,

$$Q(\mathbb{Q}_p) \cap K = q(Q(\mathbb{Q}_p) \cap K_1)q^{-1} = (W(\mathbb{Q}_p) \cap K) \rtimes (qL(\mathbb{Q}_p)q^{-1} \cap K),$$

we still have $\mathcal{Q} = \mathcal{W} \rtimes \mathcal{L}$, and $\mathcal{W}$ is smooth with connected fibers since it is conjugated to the one cut out by K_1. But in this case $\mathcal{L}$ is no longer the closure of L in $\mathcal{G}$, we need to conjugate the section $L \to Q$ by q. When we study the levels of mixed Shimura varieties coming from boundary, this is exactly what we do: we define the level $K_{\Phi,L} \subset L_\Phi(\mathbb{A}_f)$ as the image of $K_{\Phi,Q} \subset Q_\Phi(\mathbb{A}_f)$ under the projection $Q_\Phi \to L_\Phi$, and we do not prescribe (and do not need) a section $L_\Phi \to Q_\Phi$.

For the second statement (2), in the arguments above Definition 1.21, we use K_1 in place of K, and note that the conjugation by q shifts apartments but does not change the correspondence between x_h and x_L.

For the last statement, when K_p and Q are in good position, fix a splitting $W = U \times V$ where both U and V are products of root groups, by Bruhat-Tits theory, we have $\mathcal{W} = \mathcal{U} \times \mathcal{V}$, where both $\mathcal{U}$ and $\mathcal{V}$ are the closures of U and V in $\mathcal{W}$ respectively. $\mathcal{W}$, $\mathcal{U}$ and $\mathcal{V}$ are affine smooth group schemes with connected fibers. In general, since U is the center of W, then Q normalizes U, as in second paragraph one can conjugate $1 \to \mathcal{U} \to \mathcal{W} \to \mathcal{V} \to 1$ by elements in Q and still get the wanted exact sequence. ☐

Lemma 3.4. *Let $x \in A_{\text{red}}(G, T)$, $T \subset Q \subset G$, $\breve{K}^\circ = \mathcal{G}_x^\circ(\breve{\mathbb{Z}}_p)$, $\breve{K} = \mathcal{G}_x(\breve{\mathbb{Z}}_p)$. Let $\breve{K}_1$ be a quasi-parahoric group controlled by x, i.e., $\breve{K}^\circ \subset \breve{K}_1 \subset \breve{K}$. Then*

$$Q(\breve{\mathbb{Q}}_p) \cap \breve{K}_1 = (W(\breve{\mathbb{Q}}_p) \cap \breve{K}_1) \rtimes (L(\breve{\mathbb{Q}}_p) \cap \breve{K}_1).$$

Proof. We omit the notation $(\breve{\mathbb{Q}}_p)$. Since

$$Q \cap \breve{K}^\circ = (W \cap \breve{K}^\circ) \rtimes (L \cap \breve{K}^\circ), \quad Q \cap \breve{K} = (W \cap \breve{K}) \rtimes (L \cap \breve{K}),$$

and $W \cap \breve{K}^\circ = W \cap \breve{K}$ (by the triviality of $\pi_1(W)$), we have $W \cap \breve{K}_1 = W \cap \breve{K}^\circ$. On the other hand,

$$(Q \cap \breve{K}_1)/(Q \cap \breve{K}^\circ) = \pi(\breve{K}_1)/\pi(\breve{K}^\circ), \quad (Q \cap \breve{K})/(Q \cap \breve{K}_1) = \pi(\breve{K})/\pi(\breve{K}_1).$$

Here $\pi : Q \to L$ is the projection. Since $\pi(\breve{K})/\pi(\breve{K}^\circ) = (L \cap \breve{K})/(L \cap \breve{K}^\circ) \subset \pi_1(L)_{I,\mathrm{tor}}$ is finite abelian, there is a unique subgroup of $Q \cap \breve{K}$ containing $Q \cap \breve{K}^\circ$ with image $\pi(\breve{K}_1)/\pi(\breve{K}^\circ)$ in $\pi(\breve{K})/\pi(\breve{K}^\circ)$, this forces $Q \cap \breve{K}_1 = (W \cap \breve{K}_1) \rtimes (L \cap \breve{K}_1)$ and $L \cap \breve{K}_1 = \pi(\breve{K}_1)$. ☐

This proof also shows the following.

Lemma 3.5. *Let $\mathcal{P}_x = \mathcal{U} \rtimes \mathcal{G}_x$ be a stabilizer quasi-parahoric group scheme, and $\mathcal{P}_x^\circ = \mathcal{U} \rtimes \mathcal{G}_x^\circ$. Let $\breve{K} = \mathcal{P}_x(\breve{\mathbb{Z}}_p)$, $\breve{K}_G = \mathcal{G}_x(\breve{\mathbb{Z}}_p)$, $\breve{K}^\circ = \mathcal{P}_x^\circ(\breve{\mathbb{Z}}_p)$, $\breve{K}_G^\circ = \mathcal{G}_x^\circ(\breve{\mathbb{Z}}_p)$. Given a σ-invariant closed subgroup $\breve{K}_1 \subset \breve{K}$ that contains $\breve{K}^\circ$, let $\breve{K}_{1,G} \subset G(\breve{\mathbb{Z}}_p)$ be the image of $\breve{K}_1$, then $\breve{K}_{1,G}$ and $\breve{K}_1$ are quasi-parahoric and there exist unique quasi-parahoric models $\mathcal{P}_1 = \mathcal{U} \rtimes \mathcal{G}_1$ with $\mathcal{P}_1(\breve{\mathbb{Z}}_p) = \breve{K}_1$, $\mathcal{G}_1(\breve{\mathbb{Z}}_p) = \breve{K}_{1,G}$.*

Lemma 3.6. *Let $P_1 = U_1 \rtimes G_1$, $P_2 = U_2 \rtimes G_2$, $P_1 \to P_2$ be an embedding that is compatible with $U_1 \rtimes G_1 \to U_2 \rtimes G_2$ in the sense of 1.33. Let $\mathcal{P}_i = \mathcal{U}_i \rtimes \mathcal{G}_i$ be quasi-parahoric group schemes of P_i such that*

$$\mathcal{P}_1(\breve{\mathbb{Z}}_p) = \mathcal{P}_2(\breve{\mathbb{Z}}_p) \cap \mathcal{P}_1(\breve{\mathbb{Q}}_p), \quad \mathcal{G}_1(\breve{\mathbb{Z}}_p) = \mathcal{G}_2(\breve{\mathbb{Z}}_p) \cap \mathcal{G}_1(\breve{\mathbb{Q}}_p),$$

then the induced morphism $\mathcal{P}_1 \to \mathcal{P}_2$ is compatible with $\mathcal{G}_1 \to \mathcal{G}_2$ in the sense of 1.35.

Proof. The decomposition $\mathcal{P}_2 = \mathcal{U}_2 \rtimes \mathcal{G}_2$ fixes a section $G_2 \to P_2$. In characteristic 0, we can choose a section $G_1 \to P_1$ such that $G_1 \to P_1 \to P_2$ factors through G_2. By assumption,

$$\mathcal{G}_1(\breve{\mathbb{Z}}_p) = G_1(\breve{\mathbb{Q}}_p) \cap \mathcal{G}_2(\breve{\mathbb{Z}}_p) = G_1(\breve{\mathbb{Q}}_p) \cap \mathcal{P}_2(\breve{\mathbb{Z}}_p) = G_1(\breve{\mathbb{Q}}_p) \cap \mathcal{P}_1(\breve{\mathbb{Z}}_p),$$

this gives a section $\mathcal{G}_1 \to \mathcal{P}_1$ that induces $\mathcal{P}_1 = \mathcal{U}_1 \rtimes \mathcal{G}_1$. ☐

3.1.3. *Associating quasi-parahoric group schemes.* Recall:

$$G(\breve{\mathbb{Q}}_p)^0 = \ker(G(\breve{\mathbb{Q}}_p) \xrightarrow{\tilde{\kappa}_G} \pi_1(G)_I), \quad G(\breve{\mathbb{Q}}_p)^1 = \ker(G(\breve{\mathbb{Q}}_p) \xrightarrow{\tilde{\nu}_G} \pi_1(G)_I \otimes \mathbb{Q}),$$

Given any quasi-parahoric group scheme $\mathcal{G}$ over $\mathbb{Z}_p$, one can find $x \in \mathcal{B}_{\mathrm{red}}(G, \mathbb{Q}_p)$ such that

$$\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \cap G(\breve{\mathbb{Q}}_p)^0 \subset \mathcal{G}(\breve{\mathbb{Z}}_p) \subset \mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \cap G(\breve{\mathbb{Q}}_p)^1, \tag{3.1}$$

moreover, x is in the generic position of the facet $\mathcal{F} \subset \mathcal{B}_{\mathrm{red}}(G, \breve{\mathbb{Q}}_p)$ determined by the parahoric group scheme $\mathcal{G}^\circ$. We have $\mathcal{G}^\circ(\breve{\mathbb{Z}}_p) = \mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \cap G(\breve{\mathbb{Q}}_p)^0$.

Also, recall that, if $G \to G'$ is a surjection with a central kernel, there is a canonical $(G(\breve{\mathbb{Q}}_p) \to G'(\breve{\mathbb{Q}}_p))$-equivariant bijection $\mathcal{B}_{\mathrm{red}}(G, \breve{\mathbb{Q}}_p) \to \mathcal{B}_{\mathrm{red}}(G', \breve{\mathbb{Q}}_p)$. Here we use reduced buildings instead of extended buildings.

Lemma 3.7. *Let G be a reductive group, $Z \subset G$ be a central torus, and $G' = G/Z$. Then the surjection $G(\breve{\mathbb{Q}}_p) \to G'(\breve{\mathbb{Q}}_p)$ induces a surjection $G(\breve{\mathbb{Q}}_p)^0 \to G'(\breve{\mathbb{Q}}_p)^0$, a surjection $\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \to \mathrm{Stab}_{G'(\breve{\mathbb{Q}}_p)}(x')$, and a morphism $G(\breve{\mathbb{Q}}_p)^1 \to G'(\breve{\mathbb{Q}}_p)^1$ where $x' \in \mathcal{B}_{\mathrm{red}}(G', \mathbb{Q}_p)$ is the image of some $x \in \mathcal{B}_{\mathrm{red}}(G, \mathbb{Q}_p)$. In particular, given a quasi-parahoric (resp. parahoric) group scheme $\mathcal{G}$ of G associated with a point $x \in \mathcal{B}_{\mathrm{red}}(G, \mathbb{Q}_p)$ in the sense of (3.1), then the image of $\mathcal{G}(\breve{\mathbb{Z}}_p)$ contains (resp. is) the parahoric group $(\mathcal{G}'_{x'})^\circ(\breve{\mathbb{Z}}_p)$ and is contained in the stabilizer quasi-parahoric group $\mathcal{G}'_{x'}(\breve{\mathbb{Z}}_p)$.*

Proof. Since Z is connected, $H^1(\breve{\mathbb{Q}}_p, Z)$ is trivial by Steinberg's theorem, $G(\breve{\mathbb{Q}}_p) \to G'(\breve{\mathbb{Q}}_p)$ is a surjection. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & Z(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "\tilde{\kappa}_Z"] & G(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "\tilde{\kappa}_G"] & G'(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "\tilde{\kappa}_{G'}"] & 1 \\
 & \pi_1(Z)_I \arrow[r] & \pi_1(G)_I \arrow[r] & \pi_1(G')_I \arrow[r] & 1,
\end{tikzcd}$$

here vertical maps are Kottwitz maps and are surjective ([Kot97, §7.1]). Chasing the diagram, we have a surjective $G(\breve{\mathbb{Q}}_p)^0 \to G'(\breve{\mathbb{Q}}_p)^0$.

Let $g \in \mathrm{Stab}_{G'(\breve{\mathbb{Q}}_p)}(x')$ and $g' \in G(\breve{\mathbb{Q}}_p)$ be a point in its preimage. Since $\mathcal{B}_{\mathrm{red}}(G, \breve{\mathbb{Q}}_p) \xrightarrow{\sim} \mathcal{B}_{\mathrm{red}}(G', \breve{\mathbb{Q}}_p)$ is $G(\breve{\mathbb{Q}}_p)$-$G'(\breve{\mathbb{Q}}_p)$-equivariant, g' fixes x; thus, $\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \to \mathrm{Stab}_{G'(\breve{\mathbb{Q}}_p)}(x')$ is surjective and moreover, $\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x)$ is the preimage of $\mathrm{Stab}_{G'(\breve{\mathbb{Q}}_p)}(x')$ under $\pi : G(\breve{\mathbb{Q}}_p) \to G'(\breve{\mathbb{Q}}_p)$. This forces

$$\pi(\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x) \cap G(\breve{\mathbb{Q}}_p)^0) = \pi(\mathrm{Stab}_{G(\breve{\mathbb{Q}}_p)}(x)) \cap \pi(G(\breve{\mathbb{Q}}_p)^0) = \mathrm{Stab}_{G'(\breve{\mathbb{Q}}_p)}(x') \cap G'(\breve{\mathbb{Q}}_p)^0$$

. □

By Lemmas 3.7 and 3.5, we make the following definitions:

Definition/Lemma 3.8. *Let $\mathcal{P} = \mathcal{U} \rtimes \mathcal{G}$ be a quasi-parahoric group scheme of P, and let $\breve{K}_p = \mathcal{P}(\breve{\mathbb{Z}}_p)$. Then $\mathcal{G}^\circ$ is the parahoric group scheme $\mathcal{G}_x^\circ = \mathcal{G}_{\mathcal{F}}$ for some $x \in \mathcal{F} \subset \mathcal{B}_{\mathrm{red}}(G, \mathbb{Q}_p)$.*

1. *Let $\breve{K}_p^c \subset P^c(\breve{\mathbb{Q}}_p)$ be the image of $\breve{K}_p$ under the projection $P \to P^c$. Then $\breve{K}_p^c$ is a quasi-parahoric group, and there is a unique quasi-parahoric group scheme $\mathcal{P}^c = \mathcal{U} \rtimes \mathcal{G}^c$ of P^c with $\mathcal{P}^c(\breve{\mathbb{Z}}_p) = \breve{K}_p^c$, where $(\mathcal{G}^c)^\circ$ is the parahoric group scheme associated with $x \in \mathcal{B}_{\mathrm{red}}(G^c, \mathbb{Q}_p)$. We call $K_p^c := \mathcal{P}^c(\mathbb{Z}_p)$ and $\mathcal{P}^c$ the quasi-parahoric group (scheme)* **associated with** *K_p, and $P \to P^c$ extends to $\mathcal{P} \to \mathcal{P}^c$.*
2. *In general, let $P \to P'$ be a quotient with central multiplicative kernel. Let $\mathcal{P}_x'^{,\circ} = \mathcal{U} \rtimes \mathcal{G}_x'^{,\circ}$ (resp. $\mathcal{P}'_x = \mathcal{U} \rtimes \mathcal{G}'_x$) be the parahoric (resp. stabilizer quasi-parahoric) group scheme associated with $x \in \mathcal{B}_{\mathrm{red}}(G', \mathbb{Q}_p)$. Let $\breve{K}'_p \subset \mathcal{P}'_x(\breve{\mathbb{Z}}_p)$ be a σ-invariant subgroup that contains $\mathrm{Im}(\breve{K}_p)$ and $\mathcal{P}_x'^{,\circ}(\breve{\mathbb{Z}}_p)$ and that stabilizes $\mathcal{U}(\breve{\mathbb{Z}}_p)$. Then it is a quasi-parahoric group, and there is a unique quasi-parahoric group scheme $\mathcal{P}' = \mathcal{U} \rtimes \mathcal{G}'$ with $\mathcal{P}'(\breve{\mathbb{Z}}_p) = \breve{K}'_p$ and $(\mathcal{G}')^\circ = \mathcal{G}_x'^{,\circ}$; moreover, $P \to P'$ extends to $\mathcal{P} \to \mathcal{P}'$.*

Remark 3.9. *In fact, the subgroup generated by $\mathrm{Im}(\breve{K}_p)$ and $\mathcal{P}'^{,\circ}(\breve{\mathbb{Z}}_p)$ is just $\mathrm{Im}(\breve{K}_p) \cdot \mathcal{P}'^{,\circ}(\breve{\mathbb{Z}}_p)$ and is open compact: $\mathrm{Im}(\breve{K}_p) \subset \mathcal{P}_x'^{,\circ}(\breve{\mathbb{Z}}_p)$, thus its conjugation stabilizes $\mathcal{P}'^{,\circ}(\breve{\mathbb{Z}}_p)$.*

Now let (G, X) be a Shimura datum, and let Φ be a cusp label representative.

Definition 3.10. *Let Y_Φ be a connected subgroup satisfying $P_\Phi \subset Y_\Phi \subset ZP_\Phi$. Denote $Y_\Phi^* := Z(G)_{ac} \cdot Y_\Phi / Z(G)_{ac}$.*

By definition, we have a central isogeny $Y_\Phi^c \to Y_\Phi^* \subset G^c$; in addition, $ZP_\Phi^* = ZP_\Phi^c$.

Note that P_Φ^* can also be obtained as follows. Pick $\Phi \in \mathcal{CLR}(G, X)$. This cusp label representative maps to a cusp label representative $\Phi^* \in \mathcal{CLR}(G^c, X^c)$. Then $P_\Phi^* = P_{\Phi^*}$.

Definition 3.11. *Let $\breve{K}_{\Phi,p}^* = \breve{K}_p^c \cap P_\Phi^*(\breve{\mathbb{Q}}_p)$ and let $\mathcal{P}_\Phi^*$ be the quasi-parahoric group scheme whose group of $\breve{\mathbb{Z}}_p$-points is $\breve{K}_{\Phi,p}^*$ (see Lemma 3.3). The group $\breve{K}_p^c = \mathcal{G}^c(\breve{\mathbb{Z}}_p)$ is defined in 3.8, viewing G here as the P there.*

Then there is, by definition, a map $\breve{K}_{\Phi,p}^c \to \breve{K}_{\Phi,p}^*$; the first group is defined by Definition 3.8 with input $(P, \breve{K}_p) = (P_\Phi, \breve{K}_{\Phi,p})$. We have $\mathcal{P}_\Phi^c \to \mathcal{P}_\Phi^*$.

Definition 3.12. *The mixed Shimura datum (P_Φ, D_Φ) and the map $P_\Phi \to P_\Phi^*$ induce a mixed Shimura datum (P_Φ^*, D_Φ^*). We denote the corresponding tower with respect to the weight filtration by $(P_\Phi^*, D_\Phi^*) \to (\overline{P}_\Phi^*, \overline{D}_\Phi^*) \to (G_{\Phi,h}^*, D_{\Phi,h}^*)$. Now, the images of $\breve{K}_{\Phi,p}^*$ in $\overline{P}_\Phi^*(\breve{\mathbb{Q}}_p)$ and $G_{\Phi,h}^*(\breve{\mathbb{Q}}_p)$ are denoted by $\overline{\breve{K}}_{\Phi,p}^*$ and $\breve{K}_{\Phi,h,p}^*$, respectively.*

By Lemma 3.3, we have quasi-parahoric group schemes $\overline{\mathcal{P}}^*_\Phi$ and $\mathcal{G}^*_{\Phi,h}$ such that

$$\mathcal{P}^*_\Phi = \mathcal{W}_\Phi \rtimes \mathcal{G}^*_{\Phi,h}, \quad \overline{\mathcal{P}}^*_\Phi = \mathcal{V}_\Phi \rtimes \mathcal{G}^*_{\Phi,h}, \quad \mathcal{G}^*_{\Phi,h}(\breve{\mathbb{Z}}_p) = \breve{K}^*_{\Phi,h,p}, \quad \overline{\mathcal{P}}^*_\Phi(\breve{\mathbb{Z}}_p) = \overline{\breve{K}}^*_{\Phi,p}.$$

Moreover, the natural morphisms $\overline{P}^c_\Phi \to \overline{P}^*_\Phi$, $G^c_{\Phi,h} \to G^*_{\Phi,h}$ induce morphisms of quasi-parahoric group schemes $\overline{\mathcal{P}}^c_\Phi \to \overline{\mathcal{P}}^*_\Phi$, $\mathcal{G}^c_{\Phi,h} \to \mathcal{G}^*_{\Phi,h}$. We denote $K^*_{\Phi,h,p} = \mathcal{G}^*_{\Phi,h}(\mathbb{Z}_p)$, $\overline{K}^*_{\Phi,p} = \overline{\mathcal{P}}^*_\Phi(\mathbb{Z}_p)$ as usual.

3.2. **Mixed Shimura varieties.** In this subsection, we quickly recall some facts about mixed Shimura varieties.

Let $(P, \mathcal{X})$ be a mixed Shimura datum in the sense of [Pin90]. According to the weight filtration, there is a tower of mixed Shimura data $(P, \mathcal{X}) \to (\overline{P}, \overline{\mathcal{X}}) \to (G_h, X_h)$, where G_h is the Levi quotient of P and $\overline{P} = P/U$, where U is a normal subgroup contained in the unipotent radical of P.

Let K be a neat open compact subgroup of $P(\mathbb{A}_f)$. Define $\overline{K}$ (resp. K_h) as the image of K in $\overline{P}(\mathbb{A}_f)$ (resp. $P_h(\mathbb{A}_f)$). Thus, we have morphisms between mixed Shimura varieties $\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}}) \to \mathrm{Sh}_{K_h}(G_h, \mathcal{X}_h)$. The reflex fields of these mixed Shimura varieties are the same.

If we assume that $(P, \mathcal{X}) = (P_\Phi, D_\Phi)$ comes from a cusp label representative $\Phi \in \mathcal{CLR}(G, X)$ of a Shimura datum (G, X), we call such mixed Shimura data/varieties the boundary mixed Shimura data/varieties. See [Wu25, §1.1.3]. In this case, U is the center of the unipotent radical of P. Note that there exist mixed Shimura varieties that do not arise in this manner.

Write the cusp label representative Φ as $(Q_\Phi, X^+_\Phi, g_\Phi)$. Consider an open immersion:

$$U(\Phi) = P_\Phi(\mathbb{Q})_+ \backslash X^+_\Phi \times P(\mathbb{A}_f)/K_\Phi \subset \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)(\mathbb{C}). \tag{3.2}$$

Let

$$\beta_\Phi : U(\Phi) \to \mathrm{Sh}_K(G, X)(\mathbb{C}), \quad [(x, p)] \mapsto [(x, pg)].$$

By varying Φ, we obtain a covering of $\mathrm{Sh}_K(G, X)(\mathbb{C})$ via β_Φ. The (analytic) construction of $\mathrm{Sh}^\Sigma_K(G, X)(\mathbb{C})$ uses compactifications of $U(\Phi)$.

Note that we can view β_Φ as the composition of a natural projection

$$p : P_\Phi(\mathbb{Q})_+ \backslash X^+_\Phi \times P_\Phi(\mathbb{A}_f)/K_\Phi \to G(\mathbb{Q})_+ \backslash X^+_\Phi \times G(\mathbb{A}_f)/g_\Phi K g_\Phi^{-1} \tag{3.3}$$

followed by a right action

$$[\cdot g_\Phi] : G(\mathbb{Q})_+ \backslash X^+_\Phi \times G(\mathbb{A}_f)/g_\Phi K g_\Phi^{-1} \to G(\mathbb{Q})_+ \backslash X^+_\Phi \times G(\mathbb{A}_f)/K. \tag{3.4}$$

Given $x \in X$, denote by μ_x the Hodge cocharacter at x. From now on, fix homomorphisms h_0 and h_∞ from $\mathbb{S}$ to H_0 such that their pre-compositions with $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$, sending z to $(z, 1)$, are identical after conjugation by a matrix $\mathbf{c} \in \mathrm{GL}_2(\mathbb{C})$; the choice of $\mathbf{c}$ depends only on the choice of h_0 and h_∞, but not on the value of z. See, for example, [Pin90, 4.3] or the beginning of [Pin92, 3.6].

Therefore, $\mathrm{int}(\mathbf{c}) \circ \mu_x = u_x^{Q_\Phi} \circ h_\infty \circ \mu$ factors through $P_{\Phi,\mathbb{C}}$. We denote $\mu_{\Phi,x} := \mathrm{int}(\mathbf{c}) \circ \mu_x$ and $\mu_{\Phi,h,x} : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu_{\Phi,x}} P_{\Phi,\mathbb{C}} \to G_{\Phi,h}$. The assignment $\mu_x \mapsto \mu_{\Phi,x}$ is well defined.

We have the Borel embedding $X \hookrightarrow \check{X} \cong G/P_{\mu_x}(\mathbb{C})$ sending a point x to the Hodge filtration defined by μ_x. Here, P_* denotes the subgroup determined by the cocharacter $*$.

We then have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, hook] & x \in X^+ \arrow[l] \arrow[r, "\tau"] & D_\Phi \arrow[r] \arrow[d, hook] & D_{\Phi,h} := W_\Phi(\mathbb{R})U_\Phi(\mathbb{C})\backslash D_\Phi \arrow[d, hook] \\
\check{X} \cong G/P_{\mu_x}(\mathbb{C}) & & \check{D}_\Phi \cong P_\Phi/P_{\mu_{\Phi,x}}(\mathbb{C}) \arrow[ll, "\mathbf{c}^{-1}(-)\mathbf{c}"'] \arrow[r] & \check{D}_{\Phi,h} \cong G_{\Phi,h}/P_{\mu_{\Phi,h,x}}(\mathbb{C}).
\end{tikzcd}$$

3.3. **p-adic local systems.** Fix a mixed Shimura datum $(P, \mathcal{X})$ in this subsection. The subgroup $K \subset P(\mathbb{A}_f)$ is assumed to be a neat open compact subgroup. If we assume that $K = K_p K^p$, then this always means that K_p is open compact in $P(\mathbb{Q}_p)$ and $K^p \subset P(\mathbb{A}_f^p)$ is neat open compact. Sometimes, we assume that $(P, \mathcal{X}) = (G, X)$ is a Shimura datum and consider boundary mixed Shimura data.

3.3.1. Let $(P, \mathcal{X})$ be a mixed Shimura datum. By [Pin90, Lem. 3.7(b)], for compact subgroups $K' \subset K \subset P(\mathbb{A}_f)$ with $K = K_p K^p \subset P(\mathbb{A}_f)$ neat open compact, we have

$$\varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(P, \mathcal{X}) = \varprojlim_{K' \subset K} P(\mathbb{Q}) \backslash \mathcal{X} \times P(\mathbb{A}_f)/K' = \varprojlim_{K' \subset K} P(\mathbb{Q}) \backslash \mathcal{X} \times P(\mathbb{A}_f)/Z(P)(\mathbb{Q})^- K',$$

where $Z(P)(\mathbb{Q})^-$ is the closure of $Z(P)(\mathbb{Q})^\circ := \{z \in Z(P)(\mathbb{Q}) | z|_{\mathcal{X}} = \mathrm{id}\}$ in $Z(P)(\mathbb{A}_f)$. Similarly, we denote by $Z(P)(\mathbb{Q})_K^-$ the closure of $Z(P)(\mathbb{Q})^\circ \cap K$ in K. We let K' run over the neat open compact subgroups in the form $K' = K_p' K^p$. Taking the inverse limit induces a natural pro-étale torsor

$$\varprojlim_{K_p' \subset K_p} \mathrm{Sh}_{K_p' K^p}(P, \mathcal{X}) \to \mathrm{Sh}_K(P, \mathcal{X}) \tag{3.5}$$

under the group $\mathscr{G}_p(P) := K/Z(P)(\mathbb{Q})_K^- K^p$.

Definition 3.13. *Let $K_p = \mathcal{P}(\mathbb{Z}_p)$ be a quasi-parahoric subgroup and define*

$$\mathbb{P}_K(P, \mathcal{X}) := \varprojlim_{K_p' \subset K_p} \mathrm{Sh}_{K_p' K^p}(P, \mathcal{X}) \times^{\underline{K/Z(P)(\mathbb{Q})_K^- K^p}} \underline{K_p^c},$$

where $K_p^c = \mathcal{P}^c(\mathbb{Z}_p)$ is the one defined by Definition 3.8 for (P, K_p).

Note that $Z(P)_K^- = Z(P)_{ac}(\mathbb{Q})^- \cap K$ as K is neat, the quotient map $K \to K_p^c$ factors through $K/Z(P)(\mathbb{Q})_K^- K^p$.

In the situation where $(P, \mathcal{X}) = (G, X)$ is a Shimura datum, fix a cusp label representative $\Phi = (Q_\Phi, X_\Phi^+, g_\Phi) \in \mathcal{CLR}(G, X)$. Then, in this context, (3.5) defines a pro-étale torsor

$$\varprojlim_{K_{\Phi,p}' \subset K_{\Phi,p}} \mathrm{Sh}_{K_{\Phi,p}' K_\Phi^p}(P_\Phi, D_\Phi) \to \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$$

under $K_\Phi / Z(P_\Phi)(\mathbb{Q})_{K_\Phi}^- K_\Phi^p$. Let $\mathcal{P}_\Phi$ be the smoothing of the closure of P_Φ in $\mathcal{G}_\Phi$, where $\mathcal{G}_\Phi$ is the conjugate of $\mathcal{G}$ by $g = g_{\Phi,p} \in G(\mathbb{Q}_p)$, the p-factor of g_Φ. Let $\mathcal{P}_\Phi^c$ be the quasi-parahoric group scheme associated with $\mathcal{P}_\Phi$.

From Definition 3.13, we can construct a pro-étale torsor $\mathbb{P}_{K_\Phi} \to \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$ under $\underline{\mathcal{P}_\Phi^c(\mathbb{Z}_p)}$. Note that we can further push out the torsor to $\mathcal{P}_\Phi^*$; we will in fact do this in the final step.

3.3.2. Let us go back to the general setup. Let $\rho : P(\mathbb{Q}_p) \to P^c(\mathbb{Q}_p) \to \mathrm{GL}(W_{\mathbb{Q}_p})$ be a finite-dimensional $\mathbb{Q}_p$-representation. Since K_p is compact, there exists a $\mathbb{Z}_p$-lattice $W_{\mathbb{Z}_p} \subset W_{\mathbb{Q}_p}$ such that $\rho(\mathcal{P}(\mathbb{Z}_p)) \subset \mathrm{GL}(W_{\mathbb{Z}_p})$.

We construct $\mathbb{L}_{\rho, W_{\mathbb{Q}_p}}$ as follows: Let

$$K_p^{(n)} = K_p \cap \rho^{-1}(\{g \in \mathrm{GL}(W_{\mathbb{Z}_p}) |\ g \equiv \mathrm{id} \mod p^n\}).$$

We then have an étale $\mathbb{Z}_p/p^n\mathbb{Z}_p$-local system $\mathbb{L}_{\rho, W_{\mathbb{Z}_p}, n}$ on Sh_K defined as

$$\mathrm{Sh}_{K_p^{(n)} K^p} \times^{K_p/K_p^{(n)}} W_{\mathbb{Z}_p}/p^n, \tag{3.6}$$

and we have

$$\mathbb{L}_{\rho, W_{\mathbb{Z}_p}} = \varprojlim_n \mathbb{L}_{\rho, W_{\mathbb{Z}_p}, n}, \quad \mathbb{L}_{\rho, W_{\mathbb{Q}_p}} = \mathbb{L}_{\rho, W_{\mathbb{Z}_p}} \otimes \mathbb{Q}.$$

We say $\mathbb{P}_K := \mathbb{P}_K(P, \mathcal{X})$ is *de Rham* if, for any such pair $(\rho, W_{\mathbb{Z}_p})$, $\mathbb{L}_{\rho, W_{\mathbb{Q}_p}}$ is de Rham. It suffices to check this on a single faithful representation of $P^c(\mathbb{Q}_p)$. This is independent of the choice of $W_{\mathbb{Z}_p} \subset W_{\mathbb{Q}_p}$.

Proposition 3.14 (Liu-Zhu)**.** *The pro-étale torsor* $\mathbb{P}_K$ *under* $\underline{\mathcal{P}^c(\mathbb{Z}_p)}$ *over* $\mathrm{Sh}_K^{\mathrm{ad}}$ *is de Rham.*

Proof. By the rigidity theorem in [LZ17, Thm. 1.1], it suffices to show that, on every geometrically connected component of Sh_K, there exists a closed point where the torsor is de Rham. In [Pin90, Def. 11.5], for each morphism of mixed Shimura data $\iota : (T, Y) \to (P, \mathcal{X})$ (which is also called a special point when (T, Y) is a pure Shimura datum such that T is a torus embedded into P) and for $K_T \subset T(\mathbb{A}_f)$, $K \subset P(\mathbb{A}_f)$ such that $\iota(K_T) \subset K$, the induced morphism $\mathrm{Sh}_{K_T}(T, Y)_{\mathbb{C}} \to \mathrm{Sh}_K(P, \mathcal{X})_{\mathbb{C}}$ descends to a map between canonical models $\mathrm{Sh}_{K_T}(T, Y) \to \mathrm{Sh}_K(P, \mathcal{X}) \times_{E(P,\mathcal{X})} E(T, Y)$. Then [LZ17, Lem. 4.4] implies that $\mathbb{L}_{\rho, W_{\mathbb{Q}_p}}$ is de Rham at special points. Note that, although the notion of special points for mixed Shimura varieties is slightly generalized in the sense that (T, Y) can be a finite cover of a usual Shimura datum, the argument in *loc. cit.* goes through verbatim since it only involves class field theory. Since we have a special point on each geometrically connected component by [Pin90, Lem. 11.6] and the transitivity of the action of $P(\mathbb{A}_f)$ on geometrically connected components, the proposition follows. ☐

Replacing P by $\overline{P}$ (resp. by G_h), we have a pro-étale torsor $\overline{\mathbb{P}}_K \to \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}})$ (resp. $\mathbb{P}_{K_h} \to \mathrm{Sh}_{K_h}(G_h, \mathcal{X}_h)$) under $\underline{\overline{\mathcal{P}}^c(\mathbb{Z}_p)}$ (resp. $\underline{\mathcal{G}_h^c(\mathbb{Z}_p)}$). Since $P \to \overline{P} \to G_h$ induces $P^c \to \overline{P}^c \to G_h^c$ by Lemma 3.1, it follows directly from the construction that we have a commutative diagram

$$\begin{tikzcd}
\mathbb{P}_K \arrow[r] \arrow[d] & \overline{\mathbb{P}}_K \arrow[r] \arrow[d] & \mathbb{P}_{K_h} \arrow[d] \\
\mathrm{Sh}_K(P, \mathcal{X}) \arrow[r] & \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}}) \arrow[r] & \mathrm{Sh}_{K_h}(G_h, \mathcal{X}_h).
\end{tikzcd}$$

Exactly the same argument as Proposition 3.14 shows that $\overline{\mathbb{P}}_K$ and $\mathbb{P}_{K_h}$ are also de Rham.

3.3.3. *Hodge-Tate period maps.* Recall that the pro-étale torsor $\mathbb{P}_K$ over $\mathrm{Sh}_K(P, \mathcal{X})$ under $\underline{\mathcal{P}^c(\mathbb{Z}_p)}$ is de Rham. Then there is a $\underline{\mathcal{P}^c(\mathbb{Z}_p)}$-equivariant Hodge-Tate period map

$$\mathrm{HT}_K : \mathbb{P}_K \to \mathrm{Gr}_{P^c, \mu^{c,-1}}\,. \tag{3.7}$$

Here the superscript "$(-)^c$" means the projection to P^c. Similarly, we have

$$\mathrm{HT}_{K_h} : \mathbb{P}_{\overline{K}} \to \mathrm{Gr}_{\overline{P}^c, \overline{\mu}^{c,-1}}, \quad \mathrm{HT}_{K_h} : \mathbb{P}_{K_h} \to \mathrm{Gr}_{G_h^c, \mu_h^{c,-1}}\,.$$

Lemma 3.15. *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathbb{P}_K \arrow[r] \arrow[d, "\mathrm{HT}_K"'] & \overline{\mathbb{P}}_K \arrow[r] \arrow[d, "\mathrm{HT}_{\overline{K}}"'] & \mathbb{P}_{K_h} \arrow[d, "\mathrm{HT}_{K_h}"'] \\
\mathrm{Gr}_{P^c, \mu^{c,-1}} \arrow[r] & \mathrm{Gr}_{\overline{P}^c, \overline{\mu}^{c,-1}} \arrow[r] & \mathrm{Gr}_{G_h^c, \mu_h^{c,-1}}\,.
\end{tikzcd} \tag{3.8}$$

Proof. We prove the commutativity of the left square, and the right one follows from the same reason.

By construction, the push-out torsor $\mathbb{P}'_K := \mathbb{P}_K \times^{\underline{\mathcal{P}_\Phi^c(\mathbb{Z}_p)}} \underline{\overline{\mathcal{P}}_\Phi^c(\mathbb{Z}_p)}$ is the pullback of $\overline{\mathbb{P}}_K$ along $\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}})$. The Hodge-Tate period map of $\mathbb{P}'_K$ is given by the composition $\mathbb{P}_K \to \mathrm{Gr}_{P^c, \mu^{c,-1}} \to \mathrm{Gr}_{\overline{P}^c, \overline{\mu}^{c,-1}}$. Since the Hodge-Tate period maps are defined using the Hodge filtrations associated with the de Rham pro-étale torsors, $\mathbb{P}'_K \to \mathrm{Gr}_{\overline{P}^c, \overline{\mu}^{c,-1}}$ factors through $\overline{\mathbb{P}}_K$. ☐

By [LZ17] (cf. Proposition 3.14) and [PR24, §2.6], the de Rham local system $\mathbb{P}_K$ (resp. $\overline{\mathbb{P}}_K$ and resp. $\mathbb{P}_{K_h}$) canonically induces a shtuka in $\mathrm{Sht}_{\mathcal{P}^c,\mu^c,\mathrm{Spd}\,E}$ (resp. $\mathrm{Sht}_{\overline{\mathcal{P}}^c,\bar{\mu}^c,\mathrm{Spd}\,E}$ and resp. $\mathrm{Sht}_{\mathcal{G}_h^c,\mu_h^c,\mathrm{Spd}\,E}$) on $\mathrm{Sh}_K(P,\mathcal{X})^\lozenge$ (resp. $\mathrm{Sh}_{\overline{K}}(\overline{P},\overline{\mathcal{X}})^\lozenge$ and resp. $\mathrm{Sh}_{K_h}(G_h,\mathcal{X}_h)^\lozenge$). Together with Corollary 1.47, we know that:

Corollary 3.16. *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{Sh}_K(P,\mathcal{X})^\lozenge \arrow[r] \arrow[d] & \mathrm{Sh}_{\overline{K}}(\overline{P},\overline{\mathcal{X}})^\lozenge \arrow[r] \arrow[d] & \mathrm{Sh}_{K_h}(G_h,\mathcal{X}_h)^\lozenge \arrow[d] \\
\mathrm{Sht}_{\mathcal{P}^c,\mu^c,\delta=1,\mathrm{Spd}\,E} \arrow[r] & \mathrm{Sht}_{\overline{\mathcal{P}}^c,\bar{\mu}^c,\delta=1,\mathrm{Spd}\,E} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_h^c,\mu_h^c,\delta=1,\mathrm{Spd}\,E}.
\end{tikzcd} \tag{3.9}$$

3.3.4. *Some sections.* Let $(P,\mathcal{X})$ be a mixed Shimura datum, $1 \to W \to P_1 \to P \to 1$ be an extension where W is a unipotent group. Assume there exists a mixed Shimura datum $(P_1,\mathcal{X}_1)$ such that $(P_1,\mathcal{X}_1)/W \cong (P,\mathcal{X})$. Assume $P_1 = W \rtimes P$, we fix a section $(P,\mathcal{X}) \to (P_1,\mathcal{X}_1)$ ([Pin90, Prop. 2.17(b)]). Let $K_1 \subset P_1(\mathbb{A}_f)$ be a neat open compact subgroup such that $K_1 = K_W \rtimes K$, where $K_W = K_1 \cap W(\mathbb{A}_f) \subset W(\mathbb{A}_f)$, $K \subset P(\mathbb{A}_f)$ is the image of K_1, then the natural projection $p_K : \mathrm{Sh}_{K_1}(P_1,\mathcal{X}_1) \to \mathrm{Sh}_K(P,\mathcal{X})$ has a section $e_K : \mathrm{Sh}_K(P,\mathcal{X}) \to \mathrm{Sh}_{K_1}(P_1,\mathcal{X}_1)$. Given $K' \subset K$ such that $K_1' = K_W' \rtimes K'$, then $e_{K'}$ and $p_{K'}$ are compatible with e_K and p_K respectively. Passing to p^∞-limit, we then have a commutative diagram:

$$\begin{tikzcd}
\mathbb{P}_K \arrow[r, "e"] \arrow[d] & \mathbb{P}_{K_1} \arrow[r, "p"] \arrow[d] & \mathbb{P}_K \arrow[d] \\
\mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "e_K"] & \mathrm{Sh}_{K_1}(P_1,\mathcal{X}_1) \arrow[r, "p_K"] & \mathrm{Sh}_K(P,\mathcal{X}).
\end{tikzcd}$$

Let $\mathcal{P}$ be a quasi-parahoric model of P, and let $\mathcal{W}$ be a smooth affine model of W with connected fibers such that $\mathcal{P}$ normalizes $\mathcal{W}$. Let $\mathcal{P}_1 = \mathcal{W} \rtimes \mathcal{P}$, $K_W = \mathcal{W}(\mathbb{Z}_p)$, $K_1 = \mathcal{P}_1(\mathbb{Z}_p)$, and $K = \mathcal{P}(\mathbb{Z}_p)$; then $K_1 = K_W \rtimes K$. The fixed section $(P,\mathcal{X}) \to (P_1,\mathcal{X}_1)$ also fixes a section of the conjugacy class of Hodge cocharacters $\{\mu\} \to \{\mu_1\}$. Define $\mathcal{P}^c$, $\mathcal{P}_1^c$ as in Definition 3.8; then, by Lemma 3.1, $P_1^c = W \rtimes P^c$, and $\mathcal{P}_1^c = \mathcal{W} \rtimes \mathcal{P}^c$.

Lemma 3.17. *We have a commutative diagram:*

$$\begin{tikzcd}
\mathbb{P}_K \arrow[r] \arrow[d, "\mathrm{HT}_K"'] & \mathbb{P}_{K_1} \arrow[r] \arrow[d, "\mathrm{HT}_{K_1}"'] & \mathbb{P}_K \arrow[d, "\mathrm{HT}_K"'] \\
\mathrm{Gr}_{P^c,\mu^{c,-1}} \arrow[r] & \mathrm{Gr}_{P_1^c,\mu_1^{c,-1}} \arrow[r] & \mathrm{Gr}_{P^c,\mu^{c,-1}}
\end{tikzcd}$$

Proof. The right diagram commutes due to Lemma 3.15. For the left diagram, fix a representation $\rho : P^c(\mathbb{Q}_p) \to P_1^c(\mathbb{Q}_p) \xrightarrow{\rho_1} \mathrm{GL}(W_{\mathbb{Q}_p})$ and a lattice $W_{\mathbb{Z}_p} \subset W_{\mathbb{Q}_p}$ stabilized by $\mathcal{P}_1^c(\mathbb{Z}_p)$. The induced local system $\rho_*\mathbb{P}_K$ on $\mathrm{Sh}_K(P,\mathcal{X})$ is the pullback of the induced local system $\rho_{1*}\mathbb{P}_{K_1}$ on $\mathrm{Sh}_{K_1}(P_1,\mathcal{X}_1)$; thus, the Hodge-Tate period maps commute. □

Lemma 3.18. *We have a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "e_K"] \arrow[d] & \mathrm{Sh}_{K_1}(P_1,\mathcal{X}_1) \arrow[r, "p_K"] \arrow[d] & \mathrm{Sh}_K(P,\mathcal{X}) \arrow[d] \\
\mathrm{Sht}_{\mathcal{P}^c,\mu^c} \arrow[r] & \mathrm{Sht}_{\mathcal{P}_1^c,\mu_1^c} \arrow[r] & \mathrm{Sht}_{\mathcal{P}^c,\mu^c}.
\end{tikzcd}$$

We can apply this result to $(P_\Phi, D_\Phi) \to (\overline{P}_\Phi, \overline{D}_\Phi)$ and to $(\overline{P}_\Phi, \overline{D}_\Phi) \to (G_{\Phi,h}, D_{\Phi,h})$.

Lemma 3.19. *Assume* $K_\Phi = (K_\Phi \cap U_\Phi(\mathbb{A}_f)) \rtimes \overline{K}_\Phi$ *and* $\overline{K}_\Phi = (\overline{K}_\Phi \cap V_\Phi(\mathbb{A}_f)) \rtimes K_{\Phi,h}$. *Then the reduction (3.9) admits a section:*

$$\begin{tikzcd}
\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \arrow[d] & \mathrm{Sh}_{\overline{K}_\Phi}(\overline{P}_\Phi, \overline{D}_\Phi)^\lozenge \arrow[l] \arrow[d] & \mathrm{Sh}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^\lozenge \arrow[l] \arrow[d] \\
\mathrm{Sht}_{\mathcal{P}^c_\Phi, \mu^c_\Phi, \delta=1, \mathrm{Spd}\, E} & \mathrm{Sht}_{\overline{\mathcal{P}}^c_\Phi, \bar{\mu}^c_\Phi, \delta=1, \mathrm{Spd}\, E} \arrow[l] & \mathrm{Sht}_{\mathcal{G}^c_{\Phi,h}, \mu^c_{\Phi,h}, \delta=1, \mathrm{Spd}\, E}. \arrow[l]
\end{tikzcd}$$

3.4. **Torus action on the tower.** We re-organize the materials in [Pin90, 3.13-3.16] and [MP19, 2.1.11], and describe the torus action in an explicit and functorial way. This exposition is helpful in §3.4.3, where we compute the torus action on a tower of mixed Shimura varieties.

Throughout the subsection, we pick any mixed Shimura datum $(P, \mathcal{X})$ and denote its reflex field by $\mathbb{E}(P, \mathcal{X})$; the symbol E denotes its completion at a place v over p. Sometimes we use the shorthand $\mathbb{E} := \mathbb{E}(P, \mathcal{X})$.

3.4.1. Let $U \cong \mathbb{G}_a^n$ be a connected additive group over $\mathbb{Q}$. The group $\mathbb{G}_m$ acts on U by scalar multiplication. Denote by $(\mathbb{G}_m, \mathcal{H}_0)$ the "Siegel Shimura datum of dimension zero", where $\mathcal{H}_0 = \{\pm\mathrm{id}\}$ is the two-point set parameterizing the isomorphisms $\mathbb{Z} \cong \mathbb{Z}(1)$. Let $P_0 = U \rtimes \mathbb{G}_\mathrm{m}$, where $\mathbb{G}_\mathrm{m}$ acts on U by this scalar multiplication. We can write P_0 as a blocked mirabolic subgroup

$$\begin{pmatrix} \mathbb{G}_m & U \\ 0 & 1 \end{pmatrix}.$$

In what follows, denote $U := \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$ by abuse of notation.

Let X_0 be $\pi_0(\mathcal{H}_0)\times$

$$(P_0(\mathbb{R})U(\mathbb{C})\text{-orbit of the homomorphism } (z_1, z_2) \in \mathbb{S}(\mathbb{C}) \mapsto \mathrm{diag}\{z_1 z_2, 1\} \in P_0(\mathbb{C})).$$

Then (P_0, X_0) is a mixed Shimura datum with a natural projection $\lambda_0 : (P_0, X_0) \to (\mathbb{G}_m, \mathcal{H}_0)$ obtained by the quotient by U. See also [Pin90, 2.24].

Let K_0 be a neat open compact subgroup of $P_0(\mathbb{A}_f)$. By [Pin90, 3.13], $\mathrm{Sh}_{K_0}(P_0, X_0)(\mathbb{C})$ is a torus torsor under $\mathbf{E}_{K_0}(\mathbb{C})$, where $\mathbf{E}_{K_0}(\mathbb{C}) \cong U(\mathbb{C})/\Lambda_0$ for $\Lambda_0 := (U(\mathbb{Q}) \cap K_0)(-1)$. More precisely, fix any $p \in P_0(\mathbb{A}_f)$, the fiber of $\mathrm{Sh}_{K_0}(P_0, X_0)(\mathbb{C}) \to \mathrm{Sh}_{\lambda_0(K_0)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) := \mathbb{Q}^\times_{>0}\backslash\{1\} \times \mathbb{A}_f^\times/K_0$ at $[(1, \lambda_0(p))]$ determined by $\lambda_0(p) \in \mathbb{G}_m(\mathbb{A}_f)$ is $U(\mathbb{C})/(U(\mathbb{Q}) \cap pK_0p^{-1})(-1)$. Note that $U(\mathbb{Q}) \cap pK_0p^{-1}$ and $U(\mathbb{Q}) \cap K_0$ differ by homothety; let us explain this in the next paragraph.

Recall that there is a unique factorization of multiplicative groups $\mathbb{A}_f^\times = \mathbb{Q}^\times_{>0} \times \widehat{\mathbb{Z}}^\times$. For $g \in \mathbb{A}_f^\times$, denote by $r(g)$ the $\mathbb{Q}^\times_{>0}$-factor. Denote

$$r(p) := r \circ \lambda_0(p) = \begin{pmatrix} r(p) & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote $\Lambda_{0,\mathbb{A}_f} := U(\mathbb{A}_f) \cap K_0$. Writing

$$\mathbf{E}_{K_0}(\mathbb{C}) \cong U(\mathbb{Q})\backslash U(\mathbb{C}) \times U(\mathbb{A}_f)/\Lambda_{0,\mathbb{A}_f}, \tag{3.10}$$

the homothety is given by the left conjugate by $r(p)$:

$$\mathbf{t}(p) : U(\mathbb{Q})\backslash U(\mathbb{C}) \times U(\mathbb{A}_f)/\Lambda_{0,\mathbb{A}_f} \xrightarrow{\sim} U(\mathbb{Q})\backslash U(\mathbb{C}) \times U(\mathbb{A}_f)/p\Lambda_{0,\mathbb{A}_f}p^{-1},$$

sending $[(u, u_f)]_{\Lambda_{0,\mathbb{A}_f}} \mapsto [(r(p)ur(p)^{-1}, r(p)u_f r(p)^{-1})]_{r(p)\Lambda_{0,\mathbb{A}_f}r(p)^{-1}}$. Note that $r(p)\Lambda_{0,\mathbb{A}_f}r(p)^{-1} = p\Lambda_{0,\mathbb{A}_f}p^{-1}$. This isomorphism between the affine complex tori canonically descends to an isomorphism between the algebraic tori.

Define an action

$$\mathbb{E}_{K_0}(P_0, X_0)(\mathbb{C}) : \mathbf{E}_{K_0}(\mathbb{C}) \times \mathrm{Sh}_{K_0}(P_0, X_0)(\mathbb{C}) \to \mathrm{Sh}_{K_0}(P_0, X_0)(\mathbb{C})$$

as follows:

Fix an isomorphism sgn : $\mathcal{H}_0 \cong \{\pm 1\}$. Let $\delta(x) = \mathrm{diag}\{\mathrm{sgn}(x), 1\}$ for $x \in \mathcal{H}_0$. Picking any $p \in P_0(\mathbb{A}_f)$, set $[(u, u_f)]_{\Lambda_{0,\mathbb{A}_f}} \in \mathbf{E}_{K_0}(\mathbb{C}) \mapsto$

$$([(x,p)]_{K_0} \mapsto [(\mathrm{int}(\delta(x)r(p)ur(p)^{-1}\delta(x)^{-1})x, \delta(x)r(p)u_f r(p)^{-1}\delta(x)^{-1}p)]_{K_0}). \tag{3.11}$$

Lemma 3.20. *The map* $\mathbb{E}_{K_0}(P_0, X_0)(\mathbb{C})$ *is well defined and algebraic. Moreover, it descends to* $\mathbb{Q}$*.*

Proof. Fix any $u_0 \in \Lambda_{0,\mathbb{A}_f}$. There is $u_0' \in \Lambda_{0,\mathbb{A}_f}$ such that $pu_0'p^{-1} = r(p)u_0 r(p)^{-1}$, because the conjugation of $\widehat{\mathbb{Z}}^\times$ and $U(\mathbb{A}_f)$ stabilizes $\Lambda_{0,\mathbb{A}_f}$. Assume that $\mathrm{sgn}(x) = 1$ without loss of generality. Then the assignment $\mathbb{E}_{K_0}(P_0, X_0)(\mathbb{C})(u, u_f \cdot u_0)$ sends $[(x,p)]_{K_0}$ to

$$\begin{aligned}
&[(\mathrm{int}(r(p)ur(p)^{-1})x, r(p)u_f u_0 r(p)^{-1} \cdot p)]_{K_0} \\
&= [(\mathrm{int}(r(p)ur(p)^{-1})x, r(p)u_f r(p)^{-1} r(p)u_0 r(p)^{-1} \cdot p)]_{K_0} \\
&= [(\mathrm{int}(r(p)ur(p)^{-1})x, r(p)u_f r(p)^{-1} pu_0'p^{-1} \cdot p)]_{K_0} \\
&= [(\mathrm{int}(r(p)ur(p)^{-1})x, r(p)u_f r(p)^{-1} \cdot p)]_{K_0}.
\end{aligned}$$

It is easy to check that this map does not change when multiplying (u, u_f) to $(u'u, u'u_f)$ by $u' \in U(\mathbb{Q})$, and that the compatibility of this map with the multiplication on $\mathbf{E}_{K_0}$. So this is a well-defined action. Since $\mathbf{t}(p)$ is algebraic and since it suffices to check the algebraicity on each individual connected component, we find that $\mathbb{E}_{K_0}(P_0, X_0)(\mathbb{C})$ is algebraic.

Now we consider the Galois action. It suffices to check on a dense subset. So we may assume that $u \in U(\mathbb{Q})$, $u_f = 1$ by strong approximation, and $[(x,p)]_K$ is contained in a special point $(T, \mathcal{Y}) \subset (P_0, X_0)$ such that

$$\mathcal{Y} \subset \pi_0(\mathcal{H}_0) \times P_0(\mathbb{Q})\text{-orbit of the homomorphism } h_0 : (z_1, z_2) \in \mathbb{S}(\mathbb{C}) \mapsto \mathrm{diag}\{z_1 z_2, 1\} \in P_0(\mathbb{C}).$$

Then, by taking a conjugation in $P_0(\mathbb{Q})$, we may consider the case where $T = \mathrm{diag}\{\mathbb{G}_m, 1\}$ and $\mathcal{Y} = \{\pm h_0\}$. Again, we assume that $\mathrm{sgn}(x) = 1$ and the case of $x = -1$ is similar. Now, for $\tau \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, we denote by $[(x,p)] \mapsto [(x, r_{(T,\mathcal{Y})}(\tau)p)]$ the Galois action on the points of $\mathrm{Sh}_{K_T}(T, \mathcal{Y})(\overline{\mathbb{Q}})$ for a suitable K_T. Write $d := r_{(T,\mathcal{Y})}(\tau) \in \pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q}))$; we choose a representative of it in $\widehat{\mathbb{Z}}^\times$. Writing $u = \begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix}$, we check that $\tau \cdot [(u,1)] \cdot [(x,p)]$

$$\begin{aligned}
&= \tau \cdot [(\mathrm{int}(r(p)ur(p)^{-1})x, p)] \\
&= \tau \cdot [(x, r(p)u^{-1}r(p)^{-1} \cdot p)] \\
&= \tau \cdot [(x, \begin{pmatrix} p & -r(p)\underline{u} \\ 0 & 1 \end{pmatrix})] \\
&= [(x, \begin{pmatrix} p & -r(p)\underline{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix})] \\
&= [(x, \begin{pmatrix} dp & -r(p)\underline{u} \\ 0 & 1 \end{pmatrix})],
\end{aligned} \tag{3.12}$$

and that $[(u,1)] \cdot \tau \cdot [(x,p)]$

$$\begin{aligned}
&= [(u,1)] \cdot [(x, d \cdot p)] \\
&= [(\mathrm{int}(r(d)r(p)ur(p)^{-1}r(d)^{-1})x, d \cdot p)] \\
&= [(x, r(d)r(p)u^{-1}r(p)^{-1}r(d)^{-1}dp)] \\
&= [(x, r(p)u^{-1}r(p)^{-1}dp)] \\
&= [(x, \begin{pmatrix} dp & -r(p)\underline{u} \\ 0 & 1 \end{pmatrix})].
\end{aligned} \tag{3.13}$$

Hence, we see that (3.12)=(3.13). Note that the multiplication of d induced by the Galois action in (3.12) is on the right because one can check that this is the only way such that the Galois action determining the canonical model is compatible with right Hecke actions. By density, the Galois action commutes with the action of $\mathbf{E}_{K_0}(\mathbb{C})$ and $\mathbb{E}_{K_0}(P_0, X_0)(\mathbb{C})$ descends to $\mathbb{Q}$. ☐

Denote the algebraic action over $\mathbb{Q}$ by $\mathbb{E}_{K_0}(P_0, X_0)$.

Convention 3.21. *Note that an isomorphism* sgn *is fixed. If we change* sgn *to* $-$sgn*, the* $\mathbf{E}_{K_0}$*-action in (3.11) will be reversed. Moreover, the isomorphism (3.10) is induced by an exponential map, which also depends on a choice of* $\sqrt{-1}$*, and here we fixed* $\mathbb{C}$ *and chose* $\sqrt{-1}$ *to be* $i \in \mathbb{C}$*. Nevertheless, the torsor structure given by (3.11) is canonical up to unique isomorphism.*

From the explicit construction above, we see that

Lemma 3.22. *The morphism* $\mathbb{E}_{K_0}(P_0, X_0)$ *is functorial in* (U, K_0)*. That is, for any homomorphism* $f : U_1 \to U_2$ *between additive groups as in the beginning of §3.4.1, we produce* (P_0^1, X_0^1) *(resp.* (P_0^2, X_0^2)*) for* U_1 *(resp.* U_2*) as in the first paragraph. Then there is a morphism between mixed Shimura data* $\widetilde{f} : (P_0^1, X_0^1) \to (P_0^2, X_0^2)$ *induced by* f*. Picking neat open compact subgroups* K_0^1 *and* K_0^2 *such that* $\widetilde{f}(K_0^1) \subset K_0^2$*, we obtain a morphism between mixed Shimura varieties* $\widetilde{f} : \mathrm{Sh}_{K_0^1}(P_0^1, X_0^1) \to \mathrm{Sh}_{K_0^2}(P_0^2, X_0^2)$ *over* $\mathbb{Q}$ *that fits into the commutative diagram*

$$\begin{tikzcd}
\mathbf{E}_{K_0^1} \times \mathrm{Sh}_{K_0^1}(P_0^1, X_0^1) \arrow[r, "\mathbb{E}_{K_0^1}(P_0^1, X_0^1)"] \arrow[d, "{[f] \times \widetilde{f}}"] & \mathrm{Sh}_{K_0^1}(P_0^1, X_0^1) \arrow[d, "\widetilde{f}"] \\
\mathbf{E}_{K_0^2} \times \mathrm{Sh}_{K_0^2}(P_0^2, X_0^2) \arrow[r, "\mathbb{E}_{K_0^2}(P_0^2, X_0^2)"] & \mathrm{Sh}_{K_0^2}(P_0^2, X_0^2).
\end{tikzcd}$$

The map $[f]$ *of the homomorphism between tori is induced by* f *and exponential maps.*

Proof. This follows from the explicit formula (3.11) above. ☐

3.4.2. Now, we consider a mixed Shimura datum $(P, \mathcal{X})$. There is a weight filtration on $\mathrm{Lie}\, P$ given by the definition of $(P, \mathcal{X})$ being a mixed Shimura datum. Let U be the unipotent group whose Lie algebra corresponds to the weight-(-2)-part of $\mathrm{Lie}\, P$ and W be the unipotent radical of P. Recall that the adjoint representation of P on $\mathrm{Lie}\, U \cong U$ factors through $G_h := P/W$.

For our purpose, we assume that this action factors as

$$\mathrm{Ad} : P \xrightarrow{\lambda} \mathbb{G}_m \to \mathrm{GL}(U),$$

where $\mathbb{G}_m$ acts as scalar multiplication.

As in §3.4.1, we define $r(p) = r \circ \lambda(p)$ for $p \in P(\mathbb{A}_f)$.

Moreover, we can extend λ to a morphism between mixed Shimura data $\lambda : (P, \mathcal{X}) \to (\mathbb{G}_\mathrm{m}, \mathcal{H}_0)$; picking any connected component $\mathcal{X}^+ \subset \mathcal{X}$, this extension depends on the choice of $\mathcal{X}^+ \to \mathcal{H}_0$, which has two options.

By [Pin90, 3.13], for any neat open compact subgroup $K \subset P(\mathbb{A}_f)$, the morphism $\mathrm{Sh}_K(P, \mathcal{X})(\mathbb{C}) \to \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}})$ is a torus torsor under $\mathbf{E}_K(\mathbb{C}) := U(\mathbb{C})/\Lambda$. Here $\Lambda = [p_2(Z(P)(\mathbb{Q})^\circ \times U(\mathbb{Q}) \cap K)](-1)$, where $Z(P)(\mathbb{Q})^\circ$ is the centralizer in $Z(P)(\mathbb{Q})$ of $\mathcal{X}$.

As (3.11), we define the action $\mathbb{E}_K(P, \mathcal{X})(\mathbb{C})$ of $[(u, u_f)]_{\Lambda_{\mathbb{A}_f}}$:

$$([(x, p)]_K \mapsto [(\mathrm{int}(\delta(x)r(p)ur(p)^{-1}\delta(x)^{-1})x, \delta(x)r(p)u_f r(p)^{-1}\delta(x)^{-1}p)]_K). \tag{3.14}$$

Lemma 3.23. *The action (3.14) is well defined.*

We can write the action as a 4-step composition:

$$\begin{aligned}(3.15)\qquad & \mathbb{E}_K(P,\mathcal{X})(\mathbb{C}) : \mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X})(\mathbb{C}) = \\ & \mathbf{E}_K \times \mathrm{Sh}_{\lambda(K)}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C}) \times_{\mathrm{Sh}_{\lambda(K)}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C})} \mathrm{Sh}_K(P,\mathcal{X})(\mathbb{C}) \to \\ & \mathbf{E}_K \times \mathrm{Sh}_{K_0}(P_0,X_0)(\mathbb{C}) \times_{\mathrm{Sh}_{\lambda(K)}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C})} \mathrm{Sh}_K(P,\mathcal{X})(\mathbb{C}) \xrightarrow{\mathbb{E}_{K_0}(P_0,X_0)(\mathbb{C})} \\ & \mathrm{Sh}_{K_0}(P_0,X_0)(\mathbb{C}) \times_{\mathrm{Sh}_{\lambda(K)}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C})} \mathrm{Sh}_K(P,\mathcal{X})(\mathbb{C}) \xrightarrow{m_K(P,\mathcal{X})(\mathbb{C})} \\ & \mathrm{Sh}_K(P,\mathcal{X})(\mathbb{C}).\end{aligned}$$

Here, (P_0, X_0) is the one defined by U in §3.4.1 and $K_0 = [p_2(Z(P)(\mathbb{Q})^\circ \times U(\mathbb{A}_f) \cap K)] \rtimes \lambda(K)$. The first equality is from the definition of fiber products, the second arrow is induced by the diagonal section from $(\mathbb{G}_m, \mathcal{H}_0)$ to (P_0, X_0) by construction, and the last arrow is given by the multiplication operation in [Pin90, 2.22] *(in which it was denoted by "μ").*

Moreover, the composition $\mathbb{E}_K(P,X)(\mathbb{C})$ in (3.15) descends to an algebraic morphism over $\mathbb{E}(P,\mathcal{X})$.

Proof. The first paragraph is verified exactly the same way as Lemma 3.20. The second paragraph is proved in [Pin90, 2.22 and Cor. 3.12], combined with Lemma 3.20. The third paragraph then follows from the second sentence of Lemma 3.20, the functoriality of canonical models of mixed Shimura varieties induced by morphisms between mixed Shimura data [Pin90, Ch. 11] and the identification $(P_0, X_0) \times_{(\mathbb{G}_m,\mathcal{H}_0)} (P,\mathcal{X}) = (U \rtimes P, \mathcal{X}')$ for some $\mathcal{X}'$. ☐

We denote the algebraic morphism above by $\mathbb{E}_K(P,\mathcal{X})$.

Lemma 3.24. *The formalism of the action $\mathbb{E}_K(P,\mathcal{X})$ is functorial in $(P,\mathcal{X},K)$.*

Proof. This follows from the fact that all steps in the composition (3.15) in the second paragraph of Lemma 3.23 are functorial. See Lemma 3.22 and [Pin90, Ch. 11]. ☐

Remark 3.25. *As mentioned above, changing the choice of $\mathcal{X}^+ \to \mathcal{H}_0$ will change $\mathbb{E}_K(P,\mathcal{X})$ to $\mathbb{E}_K(P,\mathcal{X}) \circ ((-\mathrm{id}) \times \mathrm{id})$, but the $\mathbf{E}_K$-torsor structure of $\mathrm{Sh}_K(P,\mathcal{X})$ is canonical up to this isomorphism.*

3.4.3. Fix a choice of $\mathcal{X}^+ \to \mathcal{H}_0$. For any pair of neat open compact subgroups $K' \subset K$, we have a commutative diagram over the reflex field $\mathbb{E}$ by Lemma 3.24:

$$\begin{tikzcd} \mathbf{E}_{K'} \times \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[r, "\mathbb{E}_{K'}(P,\mathcal{X})"] \arrow[d] & \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[d] \\ \mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "\mathbb{E}_K(P,\mathcal{X})"] & \mathrm{Sh}_K(P,\mathcal{X}). \end{tikzcd} \tag{3.16}$$

Assume that $K = K_pK^p$. Taking the limit over all $K' = K'_pK^p \subset K$, we denote $\mathbf{E}_{\infty,K} := \varprojlim_{K'_p \subset K_p} \mathbf{E}_{K'}$. Note that the Galois group of $\mathbf{E}_{\infty,K,E}$ over $\mathbf{E}_{K,E}$ is

$$\mathscr{G}_p(\mathbf{E}_K) := \varprojlim_{K'_p \subset K_p} \frac{p_2(Z(P)(\mathbb{Q})^\circ \times U(\mathbb{Q}) \cap K_pK^p)}{p_2(Z(P)(\mathbb{Q})^\circ \times U(\mathbb{Q}) \cap K'_pK^p)}.$$

When $P = P^c$, from the neatness of K, the group $\mathscr{G}_p(\mathbf{E}_K) =$

$$\varprojlim_{K'_p \subset K_p} \frac{U(\mathbb{Q}) \cap K_pK^p}{U(\mathbb{Q}) \cap K'_pK^p} = \varprojlim_{K'_p \subset K_p} \frac{(U(\mathbb{A}_f) \cap K_pK^p) \cap U(\mathbb{Q})}{(U(\mathbb{A}_f) \cap K'_pK^p) \cap U(\mathbb{Q})}.$$

The group U is unipotent, and $U(\mathbb{A}_f) \cap K_pK^p = (U(\mathbb{Q}_p) \cap K_p)(U(\mathbb{A}_f^p) \cap K^p)$. Therefore, the group $\mathscr{G}_p(\mathbf{E}_K)$ is pro-p, and $\mathbf{E}_{\infty,K}$ is the inverse limit of $\cdots \xrightarrow{p} \mathbf{E}_K \xrightarrow{p} \mathbf{E}_K \xrightarrow{p} \mathbf{E}_K$. Taking the inverse limit of (3.16) over $K' = K'_p K^p$, we have

$$\begin{tikzcd}
\mathbf{E}_{\infty,K} \times \mathbb{P}_K \arrow[r, "\mathbb{E}_{K^p}(P,\mathcal{X})"] \arrow[d] & \mathbb{P}_K \arrow[d] \\
\mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "\mathbb{E}_K(P,\mathcal{X})"] & \mathrm{Sh}_K(P,\mathcal{X}),
\end{tikzcd} \tag{3.17}$$

where $\mathbb{E}_{K^p}(P,\mathcal{X}) = \varprojlim_{K'_p\subset K_p} \mathbb{E}_{K'}(P,\mathcal{X})$.

Now, we consider a general P. Let K^c_p be defined as in Definition 3.8, and let $K^{c,p}$ be the image of K^p in $P^c(\mathbb{A}^p_f)$. Note that K^c_p is open compact and $K^{c,p}$ is neat open compact.

By Lemma 3.24, there is a commutative diagram induced by the morphism $(P,\mathcal{X},K)\to(P^c,\mathcal{X}^c,K^c)$:

$$\begin{tikzcd}
\mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "\mathbb{E}_K(P,\mathcal{X})"] \arrow[d] & \mathrm{Sh}_K(P,\mathcal{X}) \arrow[d] \\
\mathbf{E}_{K^c} \times \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c) \arrow[r, "\mathbb{E}_{K^c}(P^c,\mathcal{X}^c)"] & \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c).
\end{tikzcd}$$

Lemma 3.26. *Taking the inverse limit with respect to all open compact subgroups $K'_p \subset K_p$, we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{E}_{\infty,K} \times \varprojlim_{K'_p\subset K_p} \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[rr, "\mathbb{E}_{K^p}(P,\mathcal{X})"] \arrow[dr] \arrow[dd] & & \varprojlim_{K'_p\subset K_p} \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[dr] \arrow[dd] & \\
& \mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X}) \arrow[rr, "\mathbb{E}_K(P,\mathcal{X})"] \arrow[dd] & & \mathrm{Sh}_K(P,\mathcal{X}) \arrow[dd] \\
\mathbf{E}_{\infty,K^c} \times \varprojlim_{K'^{,c}_p\subset K^c_p} \mathrm{Sh}_{K',c}(P^c,\mathcal{X}^c) \arrow[rr, "\mathbb{E}_{K^{c,p}}(P^c,\mathcal{X}^c)"] \arrow[dr] & & \varprojlim_{K'^{,c}_p\subset K^c_p} \mathrm{Sh}_{K',c}(P^c,\mathcal{X}^c) \arrow[dr] & \\
& \mathbf{E}_{K^c} \times \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c) \arrow[rr, "\mathbb{E}_{K^c}(P^c,\mathcal{X}^c)"] & & \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c).
\end{tikzcd}$$

Moreover, the commutative diagram

$$\begin{tikzcd}
\mathbf{E}_{\infty,K} \times \varprojlim_{K'_p\subset K_p} \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[r, "\mathbb{E}_{K^p}(P,\mathcal{X})"] \arrow[d] & \varprojlim_{K'_p\subset K_p} \mathrm{Sh}_{K'}(P,\mathcal{X}) \arrow[d] \\
\mathbf{E}_{\infty,K^c} \times \varprojlim_{K'^{,c}_p\subset K^c_p} \mathrm{Sh}_{K',c}(P^c,\mathcal{X}^c) \arrow[r, "\mathbb{E}_{K^{c,p}}(P^c,\mathcal{X}^c)"] & \varprojlim_{K'^{,c}_p\subset K^c_p} \mathrm{Sh}_{K',c}(P^c,\mathcal{X}^c)
\end{tikzcd} \tag{3.18}$$

is equivariant under the commutative diagram

$$\begin{tikzcd}
\{1\} \times K/Z(P)(\mathbb{Q})^{-}_{K} K^p \arrow[r, "\mathrm{id}"] \arrow[d] & K/Z(P)(\mathbb{Q})^{-}_{K} K^p \arrow[d] \\
\{1\} \times K^c_p \arrow[r, "\mathrm{id}"] & K^c_p
\end{tikzcd}$$

over

$$\begin{tikzcd}
\mathbf{E}_K \times \mathrm{Sh}_K(P,\mathcal{X}) \arrow[r, "\mathbb{E}_K(P,\mathcal{X})"] \arrow[d] & \mathrm{Sh}_K(P,\mathcal{X}) \arrow[d] \\
\mathbf{E}_{K^c} \times \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c) \arrow[r, "\mathbb{E}_{K^c}(P^c,\mathcal{X}^c)"] & \mathrm{Sh}_{K^c}(P^c,\mathcal{X}^c).
\end{tikzcd}$$

Proof. The first diagram follows from Lemma 3.24. Let us show the equivariance of (3.18). From the functoriality between mixed Shimura varieties, the two vertical arrows are $(K/Z(P)(\mathbb{Q})^-_K K^p \to K_p^c)$-equivariant.

We check the $(K/Z(P)(\mathbb{Q})^-_K K^p \to K/Z(P)(\mathbb{Q})^-_K K^p)$-equivariance of $\mathbb{E}_{K^p}(P, \mathcal{X})$. Fix $K'_p \subset K_p$ and assume that it is a normal subgroup. It suffices to check this over $\mathbb{C}$. Pick $[(u, 1)] \in \mathbf{E}_{K'}$ and $k \in K_p$. On the one hand, we compute that

$$[(u, 1)] \cdot k \cdot [(x, p)]_{K'} = [(u, 1)] \cdot [(x, pk)]_{K'} = [(x, \mathrm{int}(\delta(x)r(pk))(u^{-1})pk)]_{K'};$$

and, on the other hand, $k \cdot [(u, 1)] \cdot [(x, p)]_{K'} = [(x, \mathrm{int}(\delta(x)r(p))(u^{-1})pk)]_{K'}$. The two expressions are the same since $r(k) = 1$. ☐

Lemma 3.27. *With the conventions in Lemma 3.26, the pushout*

$$\mathbb{P}_K := \varprojlim_{K'_p \subset K_p} \mathrm{Sh}_{K'}(P, \mathcal{X}) \times^{\underline{K/Z(P)(\mathbb{Q})^-_K K^p}} \underline{K_p^c}$$

is the pullback of $\mathbb{P}_{K^c}$ *on* $\mathrm{Sh}_{K^c}(P^c, \mathcal{X}^c)$. *The action* $\mathbb{E}_{K^p}(P, \mathcal{X})$ *induces a* $(\mathbf{E}'_{\infty,K} \to \mathbf{E}_K)$*-equivariant action* $\mathbb{E}'_{K^p}(P, \mathcal{X})$ *on* $\mathbb{P}_K \to \mathrm{Sh}_K(P, \mathcal{X})$ *such that* $\mathbf{E}'_{\infty,K}$ *is an inverse limit of p-power covers of* $\mathbf{E}_K$*, i.e.,* $\mathbf{E}'_{\infty,K} = \varprojlim_{i \geq 0} \mathbf{E}_K^{(i)}$*, where* $\mathbf{E}_K^{(0)} = \mathbf{E}_K$ *and* $\mathbf{E}_K^{(i)} \to \mathbf{E}_K$ *are p-power isogenies between tori for positive integers* i.

In other words, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{E}'_{\infty,K} \times \mathbb{P}_K \arrow[r, "\mathbb{E}'_{K^p}(P{,}\mathcal{X})"] \arrow[d] & \mathbb{P}_K \arrow[d] \\
\mathbf{E}_K \times \mathrm{Sh}_K(P, \mathcal{X}) \arrow[r, "\mathbb{E}_K(P{,}\mathcal{X})"] & \mathrm{Sh}_K(P, \mathcal{X}).
\end{tikzcd}$$

Proof. The first sentence follows from the equivariance statement in Lemma 3.26. Fix an open compact normal subgroup $K'_p \subset K_p$. Consider the pullback of

$$\begin{tikzcd}
\mathbf{E}_{K',c} \times \mathrm{Sh}_{K',c}(P^c, \mathcal{X}^c) \arrow[r, "\mathbb{E}_{K',c}(P^c{,}\mathcal{X}^c)"] \arrow[d] & \mathrm{Sh}_{K',c}(P^c, \mathcal{X}^c) \arrow[d] \\
\mathbf{E}_{K^c} \times \mathrm{Sh}_{K^c}(P^c, \mathcal{X}^c) \arrow[r, "\mathbb{E}_{K^c}(P^c{,}\mathcal{X}^c)"] & \mathrm{Sh}_{K^c}(P^c, \mathcal{X}^c)
\end{tikzcd}$$

along $\mathbf{E}_K \times \mathrm{Sh}_K(P, \mathcal{X}) \to \mathbf{E}_{K^c} \times \mathrm{Sh}_{K^c}(P^c, \mathcal{X}^c)$. It suffices to show that the connected component of the fiber product $\mathbf{E}_{K,\mathbb{E}} \times_{\mathbf{E}_{K^c,\mathbb{E}}} \mathbf{E}_{K',c,\mathbb{E}}$ is a torus $\mathbf{E}$ such that the morphism $\mathbf{E} \to \mathbf{E}_{K,\mathbb{E}}$ induced by the projection to the first factor of this fiber product is a p-power isogeny. This is Lemma 3.28. ☐

Lemma 3.28. *Let* $\mathbf{E}_0$, $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *be split tori of the same rank over a field k of characteristic zero. Let* $f : \mathbf{E}_1 \to \mathbf{E}_0$ *and* $g : \mathbf{E}_2 \to \mathbf{E}_0$ *be isogenies between tori. Assume that g is a p-power isogeny. Then the identity component* $\mathbf{E}_3$ *of* $\mathbf{E}_1 \times_{f,\mathbf{E}_0,g} \mathbf{E}_2$ *is a split torus equipped with a p-power isogeny* $\mathbf{E}_3 \to \mathbf{E}_1$.

Proof. This fiber product is smooth, so irreducible components are exactly the connected components. If we know that $\mathbf{E}_3$ is a split torus, then $\mathbf{E}_3 \to \mathbf{E}_1$ is a p-power isogeny by construction. We show that $\mathbf{E}_3$ is a split torus. Write the character groups of $\mathbf{E}_i$ by X_i for $i = 0, 1, 2$. Then there are inclusions $X_0 \subset X_1 \subset X_0 \otimes \mathbb{Q}$ and $X_0 \subset X_2 \subset X_0 \otimes \mathbb{Q}$. Let $X' = X_1 \cap X_2 \supset X_0$ be the intersection of the lattices X_1 and X_2 in $X_0 \otimes \mathbb{Q}$. Let X be the abelian group generated by X_1 and X_2 in $X_0 \otimes \mathbb{Q}$.

We have a map $X_1 \oplus_{X'} X_2 \to X$ given by addition in $X_0 \otimes \mathbb{Q}$. This is surjective by the definition of X and is injective by the definition of X'. Hence, we see that $\mathbf{E} := \mathrm{Spec}\, k[X] \to \mathrm{Spec}\, k[X_1] \times_{\mathrm{Spec}\, k[X']} \mathrm{Spec}\, k[X_2]$ is an isomorphism. On the other hand, the map $\mathbf{E} \to \mathrm{Spec}\, k[X_1] \times_{\mathrm{Spec}\, k[X_0]} \mathrm{Spec}\, k[X_2]$ is a closed embedding by the separatedness of $\mathrm{Spec}\, k[X'] \to \mathrm{Spec}\, k[X_0]$, and is a morphism between varieties of the same dimension. We then conclude that $\mathbf{E} \cong \mathbf{E}_3 := (\mathbf{E}_1 \times_{\mathbf{E}_0} \mathbf{E}_2)^\circ$. ☐

3.4.4. *Action on shtukas.* Now, we work over E, the completion of $\mathbb{E}$ as the beginning of this subsection. Let $\mathcal{P}$ be a quasi-parahoric model of P and $K_p = \mathcal{P}(\mathbb{Z}_p)$. Fix a neat level $K^p \subset P(\mathbb{A}_f^p)$, and write $K = K_pK^p$. Let $E_K = \mathbf{E}_K \otimes \mathbb{Q}_p$, and $E'_{\infty,K} = \varprojlim(\cdots \xrightarrow{p} E_K \xrightarrow{p} E_K)$.

Note that $(E'_{\infty,K})^{\Diamond}$ is represented by a perfectoid space in the sense that, when base-changed to $\mathrm{Spa}(C, \mathcal{O}_C)$, $(E'_{\infty,K})^{\Diamond}_C$ is the $\Diamond$ of $E'_{\infty,K,C} \sim \varprojlim(\cdots \xrightarrow{p} E^{\mathrm{ad}}_{K,C} \xrightarrow{p} E^{\mathrm{ad}}_{K,C})$; see [BGH$^+$22, Lem. 2.14].

Let $(\mathscr{P}, \phi_{\mathscr{P}})$ be the $\mathcal{P}^c$-shtuka on $\mathrm{Sh}_K(P, \mathcal{X})^{\Diamond}$ associated with $(\mathbb{P}_K, \mathrm{HT}_K)$. Let F be a finite field extension of E or $\breve{E}$, and C be a non-archimedean perfectoid field containing F. Given $\gamma \in E_K(F)$, by taking p^∞-roots in C, we can lift γ to some $\tilde{\gamma} \in E'_{\infty,K}(C)$.

Lemma 3.29. *$\tilde{\gamma}$ induces an isomorphism of $\mathcal{P}^c$-shtukas $(\mathscr{P}, \phi_{\mathscr{P}})$ and $\gamma^*(\mathscr{P}, \phi_{\mathscr{P}})$ over $\mathrm{Sh}_K(P, \mathcal{X})_F$.*

Proof. In Lemma 3.27, we see that the $\tilde{\gamma}$-action on $\mathbb{P}_{K,C}$ is equivariant with the γ-action on $\mathrm{Sh}_K(P, \mathcal{X})_C$. This gives an isomorphism of pro-étale torsors $\tilde{\gamma}_* : \mathbb{P}_{K,C} \to \gamma^*\mathbb{P}_{K,C}$. We need to show that $\tilde{\gamma}_*$ is compatible with the Hodge–Tate period maps (i.e. $\mathrm{HT}_K(\tilde{\gamma}(x)) = \mathrm{HT}_K(x)$ for any $x \in \mathbb{P}_{K,C}$). The action $E'_{\infty,K,C} \times \mathbb{P}_{K,C} \to \mathbb{P}_{K,C}$ is the base change of $E'_{\infty,K} \times \mathbb{P}_K \to \mathbb{P}_K$. Since the action of elements in $E'_{\infty,K}$ is compatible with the Hodge–Tate period maps (see the proof of Proposition 5.8), $\tilde{\gamma}$ is compatible with the Hodge–Tate period maps. ☐

Remark 3.30. *We do not have a canonical lifting of $\gamma \in E_K$ to $E'_{\infty,K}$, and there is no equivariant action of E_K on $\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sht}_{\mathcal{P}^c,\mu^c}$. Thus, the $\mathcal{P}^c$-shtuka on $\mathrm{Sh}_K(P, \mathcal{X})$ does not descend to $\mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}})$. The reader can compare this situation with Proposition 5.8 later, where the action induces descent data.*

3.5. **Abelian scheme action on the tower.** We focus on the abelian-scheme torsor $\mathrm{Sh}_{\overline{K}} := \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}}) \to \mathrm{Sh}_{K_h} := \mathrm{Sh}_{K_h}(G_h, \mathcal{X}_h)$. Here $G_h := P/W = \overline{P}/V$, $K_h = \pi(\overline{K})$, and $\pi : \overline{P} \to G_h$ is the projection.

Recall that the fiber of $\mathrm{Sh}_{\overline{K}}(\mathbb{C}) \to \mathrm{Sh}_{K_h}(\mathbb{C})$ at $(x, q) \in \mathrm{Sh}_{K_h}(\mathbb{C})$ can be computed explicitly (see [Pin90, §3.13]). Let $e : (G_h, X_h) \to (\overline{P}, \overline{\mathcal{X}})$ be a splitting. Then the map $[w] \mapsto [w \cdot e(x), q]$ gives an isomorphism from the fiber over (x, q) to $\Gamma_{\overline{K}} \backslash V(\mathbb{R})$, where $\Gamma_{\overline{K}}$ is the image of $(\{z \in Z(\overline{P})(\mathbb{Q}) |\ z|_{\overline{\mathcal{X}}} = \mathrm{id}\} \rtimes V(\mathbb{Q})) \cap q\overline{K}q^{-1}$ under the projection $Z(\overline{P}) \times V \to V$. When we fix the away-from-p part, the embedding $\Gamma_{\overline{K}(p^n)} \to \Gamma_{\overline{K}(p^m)}$ $(m \leq n)$ has p-torsion cokernel and gives the projection

$$p_{n,m} : \Gamma_{\overline{K}(p^n)} \backslash V(\mathbb{R}) \to \Gamma_{\overline{K}(p^m)} \backslash V(\mathbb{R}). \tag{3.19}$$

By [Pin90, §3.22], $\Gamma_{\overline{K}} \backslash V(\mathbb{R}) = \Gamma_{\overline{K}} \backslash V(\mathbb{C}) / \exp(\mathrm{Fil}^0(\mathrm{Lie}(V))_{\mathbb{C}})$ is an abelian variety, and (3.19) is a p-isogeny of abelian varieties. This picture descends over the reflex field.

Let K_V be the image of $(\{z \in Z(\overline{P})(\mathbb{Q}) |\ z|_{\overline{\mathcal{X}}} = \mathrm{id}\} \rtimes V(\mathbb{A}_f)) \cap \overline{K}$ under the projection $Z(\overline{P}) \times V \to V$, and let $K_h^* \subset G_h(\mathbb{A}_f)$ be an open compact subgroup that contains K_h and normalizes K_V; let $\widetilde{K}^* = K_V \rtimes K_h^*$. By [Pin90, Cor. 3.12] and [Pin90, §10], $\mathrm{Sh}_{\overline{K}} \to \mathrm{Sh}_{K_h}$ is canonically a torsor under the family of abelian varieties $\mathrm{Sh}_{\widetilde{K}^*} \to \mathrm{Sh}_{K_h^*}$.

To simplify the calculation, we assume that K_h normalizes K_V (and we can, and will, always do this in later sections), and take $K_h^* = K_h$. In particular, for any pair of neat subgroups $\overline{K}' \subset \overline{K}$ such that K'_h normalizes K'_V, we have a commutative diagram over the reflex field:

$$\begin{tikzcd}
\mathrm{Sh}_{\widetilde{K}'} \times_{\mathrm{Sh}_{K'_h}} \mathrm{Sh}_{\overline{K}'} \arrow[r, "\mu_{\overline{K}'}"] \arrow[d] & \mathrm{Sh}_{\overline{K}'} \arrow[d] \\
\mathrm{Sh}_{\widetilde{K}} \times_{\mathrm{Sh}_{K_h}} \mathrm{Sh}_{\overline{K}} \arrow[r, "\mu_{\overline{K}}"] & \mathrm{Sh}_{\overline{K}}.
\end{tikzcd} \tag{3.20}$$

For our purpose, we directly work over E rather than $\mathbb{E}$. We fix a system of normal subgroups $\overline{K}' \subset \overline{K}$ such that $\overline{K}'^p = \overline{K}^p$; then $\widetilde{K}' \subset \widetilde{K}$ are normal and $\widetilde{K}'^p = \widetilde{K}^p$. Fix $x := x_{\overline{K}} \in \mathrm{Sh}_{K_h}(F)$, where F is a finite field extension of E or $\breve{E}$. Lift x to $\tilde{x} \in \mathrm{Sh}_{K_h^p}(\overline{\mathbb{Q}}_p) := \varprojlim_{K'_{h,p} \subset K_{h,p}} \mathrm{Sh}_{K'_{h,p}K_h^p}(\overline{\mathbb{Q}}_p)$, and let $x_{\overline{K}'} \in \mathrm{Sh}_{K'_h}(\overline{\mathbb{Q}}_p)$ be its image. Denote the abelian scheme $\mathrm{Sh}_{\widetilde{K}'} \to \mathrm{Sh}_{K'_h}$ by $A_{\overline{K}'} \to \mathrm{Sh}_{K'_h}$, the fiber at $x_{\overline{K}'}$ by $A_{x_{\overline{K}'}}$, and $A_{x_{\overline{K}}}$ by A_x. Also denote the fiber of $\mathrm{Sh}_{\overline{K}'} \to \mathrm{Sh}_{K'_h}$ at $x_{\overline{K}'}$ by $\mathrm{Sh}_{\overline{K}',x_{\overline{K}'}}$, and $\mathrm{Sh}_{\overline{K},x_{\overline{K}}}$ by $\mathrm{Sh}_{\overline{K},x}$. The projection $A_{\overline{K}'} \to A_{\overline{K}}$ induces a p-isogeny $A_{x_{\overline{K}'}} \to A_x$, by (3.19).

Passing to the p^∞-level, denote

$$\mathcal{S}_{\widetilde{K}^p} = \varprojlim_{\widetilde{K}'_p \subset \widetilde{K}_p} \mathrm{Sh}_{\widetilde{K}'_p\widetilde{K}^p}, \quad \mathcal{A}_{\overline{K}^p} = \varprojlim_{\overline{K}'_p \subset \overline{K}_p} A_{\overline{K}'_p\overline{K}^p}, \quad \mathcal{S}_{\widetilde{K}^p} = \mathcal{A}_{\overline{K}^p}.$$

$$\mathcal{S}_{\overline{K}^p} = \varprojlim_{\overline{K}'_p \subset \overline{K}_p} \mathrm{Sh}_{\overline{K}'_p\overline{K}^p}, \quad \mathcal{S}_{K_h^p} = \varprojlim_{K'_{h,p} \subset K_{h,p}} \mathrm{Sh}_{K'_{h,p}K_h^p}, \quad \mathcal{A}_{\infty,\tilde{x}} = \varprojlim_{\overline{K}'_p \subset \overline{K}_p} A_{x_{\overline{K}'_p\overline{K}^p}}.$$

Remark 3.31. *In abelian-type case, $\mathcal{S}^\diamondsuit_{K_h^p}$ is represented by a perfectoid space (see* [She17]*), one can show that $\mathcal{S}^\diamondsuit_{\widetilde{K}^p}$ and $\mathcal{S}^\diamondsuit_{\overline{K}^p}$ are represented by perfectoid spaces using the fact that the fibers $\mathcal{A}^\diamondsuit_{\infty,\tilde{x}}$ are represented by perfectoid spaces (see* [BGH$^+$22, Thm. 1]*). We do not need this result, so we do not spell out the proof.*

Let $\tilde{x}^c$ be the image of $\tilde{x}$ in $\mathrm{Sh}_{K_h^{c,p}}(\overline{\mathbb{Q}}_p) := \varprojlim_{K^{c,\prime}_{h,p} \subset K^c_{h,p}} \mathrm{Sh}_{K^{c,\prime}_{h,p}K_h^{c,p}}(\overline{\mathbb{Q}}_p)$. We can similarly define these objects using $\mathrm{Sh}_{\overline{K}^c}(\overline{P}^c, \overline{\mathcal{X}}^c) \to \mathrm{Sh}_{K_h^c}(G_h^c, X_h^c)$. Then diagram (3.20) induces a commutative diagram:

$$\begin{tikzcd}
A_{x_{\overline{K}'}} \times_{x_{\overline{K}'}} \mathrm{Sh}_{\overline{K}',x_{\overline{K}'}} \arrow[rr, "\mu_{x_{\overline{K}'}}"] \arrow[dr] \arrow[dd] & & \mathrm{Sh}_{\overline{K}',x_{\overline{K}'}} \arrow[dr] \arrow[dd] & \\
& A_{x_{\overline{K}^{c,\prime}}} \times_{x_{\overline{K}^{c,\prime}}} \mathrm{Sh}_{\overline{K}^{c,\prime},x_{\overline{K}^{c,\prime}}} \arrow[rr, "\mu_{x_{\overline{K}^{c,\prime}}}"] \arrow[dd] & & \mathrm{Sh}_{\overline{K}^{c,\prime},x_{\overline{K}^{c,\prime}}} \arrow[dd] \\
A_x \times_x \mathrm{Sh}_{\overline{K},x} \arrow[rr, "\mu_x"] \arrow[dr] & & \mathrm{Sh}_{\overline{K},x} \arrow[dr] & \\
& A_{x^c} \times_{x^c} \mathrm{Sh}_{\overline{K}^c,x^c} \arrow[rr, "\mu_{x^c}"] & & \mathrm{Sh}_{\overline{K}^c,x^c}.
\end{tikzcd} \tag{3.21}$$

By passing to p^∞-level, we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{A}_{\infty,\tilde{x}} \times_{\tilde{x}} \mathcal{S}_{\overline{K}^p,\tilde{x}} \arrow[rr, "\mu_{\tilde{x}}"] \arrow[dr] \arrow[dd] & & \mathcal{S}_{\overline{K}^p,\tilde{x}} \arrow[dr] \arrow[dd] & \\
& \mathcal{A}_{\infty,\tilde{x}^c} \times_{\tilde{x}^c} \mathcal{S}_{\overline{K}^{c,p},\tilde{x}^c} \arrow[rr, "\mu_{\tilde{x}^c}"] \arrow[dd] & & \mathcal{S}_{\overline{K}^{c,p},\tilde{x}^c} \arrow[dd] \\
A_x \times_x \mathrm{Sh}_{\overline{K},x} \arrow[rr, "\mu_x"] \arrow[dr] & & \mathrm{Sh}_{\overline{K},x} \arrow[dr] & \\
& A_{x^c} \times_{x^c} \mathrm{Sh}_{\overline{K}^c,x^c} \arrow[rr, "\mu_{x^c}"] & & \mathrm{Sh}_{\overline{K}^c,x^c},
\end{tikzcd} \tag{3.22}$$

where $\mathcal{S}_{\overline{K}^p,\tilde{x}}$ is the fiber of $\mathcal{S}_{\overline{K}^p} \to \mathcal{S}_{K_h^p}$ at $\tilde{x}$. Similarly for $\mathcal{S}_{\overline{K}^{c,p},\tilde{x}^c}$.

We introduce lemmas analogous to Lemma 3.27 and Lemma 3.28.

Lemma 3.32. *With the above conventions, the pushout*

$$\overline{\mathbb{P}}_K := \varprojlim_{\overline{K}'_p \subset \overline{K}_p} \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}}) \times^{\underline{\overline{K}/Z(\overline{P})(\mathbb{Q})^-_{\overline{K}}\overline{K}^p}} \underline{\overline{K}^c_p}$$

is the pullback of $\overline{\mathbb{P}}_{K^c}$ on $\mathrm{Sh}_{\overline{K}^c}(\overline{P}^c, \overline{\mathcal{X}}^c)$. The action $\mu_{\tilde{x}}$ induces a $(\mathcal{A}'_{\infty,\tilde{x}} \to A_x)$-equivariant action $\mu'_{\tilde{x}}$ on $\overline{\mathbb{P}}_{K,\tilde{x}} \to \mathrm{Sh}_{\overline{K}}(\overline{P}, \overline{\mathcal{X}})_x$, where $\mathcal{A}'_{\infty,\tilde{x}}$ is an inverse limit of p-power covers of $\mathcal{A}_x$, and $\overline{\mathbb{P}}_{K,\tilde{x}}$ is the fiber of $\overline{\mathbb{P}}_K \to \mathbb{P}_{K_h}$ at $\tilde{x}$.

Proof. Same as the proof of Lemma 3.27 (using Lemma 3.33). ☐

Lemma 3.33. *Let A_0, A_1, A_2 be abelian varieties over a field of characteristic 0, and let $f : A_1 \to A_0$ and $g : A_2 \to A_0$ be isogenies. Assume that g is a p-power isogeny. Then the identity component A_3 of $A_1 \times_{f,A_0,g} A_2$ is an abelian variety equipped with a p-power isogeny $A_3 \to A_1$.*

Proof. $A_1 \times_{f,A_0,g} A_2$ is a closed subgroup of $A_1 \times A_2$, hence projective; its identity component A_3 is therefore an abelian variety by Cartier's theorem since all varieties are defined over a field of characteristic zero. By construction, $A_3 \to A_1$ is a p-power isogeny. □

Let $(\overline{\mathscr{P}}, \phi_{\overline{\mathscr{P}}})$ be the $\overline{\mathcal{P}}^c$-shtuka on $\mathrm{Sh}^\diamond_{\overline{K}}$ associated with $(\mathbb{P}_K, \overline{\mathrm{HT}}_K)$.

Lemma 3.34. *Let $x \in \mathrm{Sh}_{K_h}(F)$. Given $\gamma \in A_x(F)$, there is an isomorphism of $\overline{\mathcal{P}}^c$-shtukas $(\overline{\mathscr{P}}, \phi_{\overline{\mathscr{P}}})$ and $\gamma^*(\overline{\mathscr{P}}, \phi_{\overline{\mathscr{P}}})$ over $\mathrm{Sh}^\diamond_{\overline{K},x}$.*

Proof. This follows similarly from the proof of Lemma 3.29. Let $\mathrm{Sh}_{\overline{K}',x}$ (resp. $A'_{\overline{K}',x}$) be the fiber of x along $\mathrm{Sh}_{\overline{K}'} \to \mathrm{Sh}_{K_h}$ (resp. $A'_{\overline{K}'} \to \mathrm{Sh}_{K'_h}$; here we use the identity component of $A_{\overline{K}} \times_{A_{\overline{K}^c}} A_{\overline{K}^{c,\prime}}$, where we pull back these abelian schemes over $\mathrm{Sh}_{K'_h}$). Let $S := \mathcal{S}_{K^p_h,x}$. Let $\mathcal{A}'_{\infty,x} = \varprojlim_{\overline{K}'_p \subset \overline{K}_p} A'_{\overline{K}'_p\overline{K}^p,x}$; it is a group object over S. Given any $x' \in S$, the fiber of $\mathcal{A}'_{\infty,x}$ at x' is $\mathcal{A}'_{\infty,\tilde{x}'}$ for some $\tilde{x}' \in \mathcal{S}_{\overline{K}^p}$ lifting x'. Consider the projection $\mathcal{A}'_{\infty,x} \to A_x \times_x S$; this is a morphism of group objects over S. Let $T_p(A_S)$ be its kernel; it is a v-sheaf over S. Recall that for any $x' \in S$, the fiber $T_p(A_S)_{x'}$ is the kernel of $\mathcal{A}'_{\infty,\tilde{x}'} \to A_x$, which is isomorphic to $T_p(A_x)$. In other words, $T_p(A_S)$ is a torsor under $T_p(A_x)_S$.

Given $\gamma \in A_x(F)$, it has image $x \in \mathrm{Sh}_{K_h}(F)$. We lift γ to $\gamma_S \in A_x(S)$. Take a v-cover $T \to S$ that trivializes $T_p(A_S)$; we can lift γ_S to $\tilde{\gamma}_T \in (\mathcal{A}'_{\infty,x})_T$ and obtain an action $(\mathcal{A}'_{\infty,x})_T \times_T (\mathbb{P}_{\overline{K},x})_T \to (\mathbb{P}_{\overline{K},x})_T$ lifting the group action $A_x \times_x \mathrm{Sh}_{\overline{K},x} \to \mathrm{Sh}_{\overline{K},x}$. In particular, the $\tilde{\gamma}_T$-action on $(\mathbb{P}_{\overline{K},x})_T$ lifts the γ-action on $\mathrm{Sh}_{\overline{K},x}$. Moreover, the action of $(\mathcal{A}'_{\infty,x})_T$ on $(\mathbb{P}_{\overline{K},x})_T$ essentially comes from the action of $\mathcal{A}'_{\infty,x}$ on $\mathbb{P}_{\overline{K},x}$ over S; the $\tilde{\gamma}_T$-action is a global lifting of γ and is compatible with the Hodge–Tate period map, thus it induces an isomorphism of $\overline{\mathcal{P}}^c$-shtukas $(\overline{\mathscr{P}}, \phi_{\overline{\mathscr{P}}})$ and $\gamma^*(\overline{\mathscr{P}}, \phi_{\overline{\mathscr{P}}})$ over $\mathrm{Sh}^\diamond_{\overline{K},x}$.

□

4. CANONICITY AND FUNCTORIALITY

4.1. **Canonical integral models.** Mimicking the axioms in [PR24], we define canonical integral models for mixed Shimura varieties coming from the boundary.

Let (G, X) be a pure Shimura datum, let $\mathcal{G}$ be a quasi-parahoric group scheme, and let $K_p = \mathcal{G}(\mathbb{Z}_p)$. We fix a cusp label $[\Phi] = [(Q_\Phi, X^+_\Phi, g_\Phi)] \in \mathrm{Cusp}_K(G, X)$. Here we can vary K^p, and do not distinguish Φ for different levels K^p once we have prescribed $g = g_\Phi \in G(\mathbb{A}_f)$. Recall that, given $\mathcal{G}$, we can attach quasi-parahoric group schemes $\mathcal{P}_\Phi$ (resp. $\mathcal{P}^c_\Phi$, $\mathcal{P}^*_\Phi$) to P_Φ (resp. P^c_Φ, P^*_Φ) as in Section 3.1.

Recall the Pappas–Rapoport integral models:

Axiom 4.1 ([PR24, Conj. 4.2.2], [DvHKZ26, Def. 4.1.2], [DY25, Def. 4.3])**.** *Consider a system $\{\mathscr{S}_K(G, X)\}_{K^p}$ of normal flat schemes over $\mathcal{O}_E$ with generic fiber $\{\mathrm{Sh}_K(G, X)\}_{K^p}$, with K^p varying over all sufficiently small compact open subgroups of $G(\mathbb{A}^p_f)$. We say $\{\mathscr{S}_K(G, X)\}_{K^p}$ is a canonical integral model of $\{\mathrm{Sh}_K(G, X)\}_{K^p}$ if the following properties are satisfied.*

(1) For every discrete valuation ring R of mixed characteristic over $\mathcal{O}_E$, we have

$$\mathrm{Sh}_K(G, X)(R[1/p]) = (\varprojlim_{K^p} \mathscr{S}_K(G, X))(R).$$

(2) *For every $K^p \subset G(\mathbb{A}^p_f)$ and $K'^p \subset G(\mathbb{A}^p_f)$ and an $h \in G(\mathbb{A}^p_f)$ with $h^{-1}K'^p h \subset K^p$, there are finite étale morphisms*

$$t_{K',K}(h) : \mathscr{S}_{K'}(G,X) \to \mathscr{S}_K(G,X)$$

extending the generic fiber.

(3) *The $\mathcal{G}^c$-shtuka $\mathscr{P}_{K,E}$ extends to a $\mathcal{G}^c$-shtuka $\mathscr{P}_K$ on $\mathscr{S}_K(G,X)^{\diamond/}$ for every sufficiently small K^p.*

(4) *Consider $x \in \mathscr{S}_K(G,X)(k)$, with corresponding $b_x \in G^c(\breve{\mathbb{Q}}_p)$. Let x_0 be the base point in $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c,b_x,\mu^c}(k)$. Then there exists an isomorphism of v-sheaves:*

$$\Theta_x : (\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c,b_x,\mu^c})^{\wedge}_{/x_0} \xrightarrow{\sim} (\mathscr{S}_K(G,X)^{\wedge}_{/x})^{\diamond},$$

such that $\Theta^*_x(\mathscr{P}_K)$ is the universal $\mathcal{G}^c$-shtuka on $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c,b_x,\mu^c}$.

We mimic the above definition:

Axiom 4.2. *Consider a system $\{\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)\}_{K^p_\Phi}$ of normal flat schemes over $\mathcal{O}_E$ with generic fiber $\{\mathrm{Sh}_{K_\Phi}(P_\Phi,D_\Phi)\}_{K^p_\Phi}$, with K^p_Φ varying over all sufficiently small compact open subgroups of $P_\Phi(\mathbb{A}^p_f)$. We say $\{\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)\}_{K^p_\Phi}$ is a canonical integral model of $\{\mathrm{Sh}_{K_\Phi}(P_\Phi,D_\Phi)\}_{K^p_\Phi}$ if the following properties are satisfied.*

(1) *For every discrete valuation ring R of mixed characteristic over $\mathcal{O}_E$, we have*

$$\mathrm{Sh}_{K_{\Phi,p}}(P_\Phi,D_\Phi)(R[1/p]) = (\varprojlim_{K^p_\Phi} \mathscr{S}_{K_{\Phi,p}K^p_\Phi}(P_\Phi,D_\Phi))(R).$$

(2) *For every $K^p_\Phi \subset P_\Phi(\mathbb{A}^p_f)$ and $K'^p_\Phi \subset P_\Phi(\mathbb{A}^p_f)$ and an $h \in P_\Phi(\mathbb{A}^p_f)$ with $h^{-1}K'^p_\Phi h \subset K^p_\Phi$, there are finite étale morphisms*

$$t_{K'_\Phi,K_\Phi}(h) : \mathscr{S}_{K'_\Phi}(P_\Phi,D_\Phi) \to \mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)$$

extending the generic fiber.

(3) *The $\mathcal{P}^*_\Phi$-shtuka $\mathscr{P}_{K_\Phi,E}$ extends to a $\mathcal{P}^*_\Phi$-shtuka $\mathscr{P}_{K_\Phi}$ on $\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)^{\diamond/}$ for every sufficiently small K^p_Φ.*

(4) *Consider $x \in \mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)(k)$, with corresponding $b_{\Phi,x} \in P^*_\Phi(\breve{\mathbb{Q}}_p) \hookrightarrow G^c(\breve{\mathbb{Q}}_p)$ (see the first paragraph of subsection 7.1). Let x_0 be the base point in $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_\Phi,b_{\Phi,x},\mu^c_\Phi}(k)$. Then there exists an isomorphism of v-sheaves:*

$$\Theta_x : (\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_\Phi,b_{\Phi,x},\mu^c_\Phi})^{\wedge}_{/x_0} \xrightarrow{\sim} (\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)^{\wedge}_{/x})^{\diamond},$$

such that $\mathcal{G}^c_\Phi \times^{\mathcal{P}^*_\Phi} \Theta^*_x(\mathscr{P}_{K_\Phi})$ is the universal $\mathcal{G}^c_\Phi$-shtuka on $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_\Phi,b_{\Phi,x},\mu^c_\Phi}$.

In general, let Y_Φ be a connected subgroup of G satisfying $P_\Phi \subset Y_\Phi \subset ZP_\Phi$, define Y^*_Φ as in Definition 3.10, we set up the axioms for the canonical integral model $\{\mathscr{S}_{K^Y_\Phi}(Y_\Phi,D^Y_\Phi)\}_{K^{Y,p}_\Phi}$ of $\{\mathrm{Sh}_{K^Y_\Phi}(Y_\Phi,D^Y_\Phi)\}_{K^{Y,p}_\Phi}$ in exactly same way.

Remark 4.3. *We use the convention that when $(P_\Phi,D_\Phi) = (G,X)$, $P^c_\Phi = P^*_\Phi = G^c$ and $\mathcal{P}^c_\Phi = \mathcal{P}^*_\Phi = \mathcal{G}^c$, so Axiom 4.2 generalizes Axiom 4.1.*

Remark 4.4. *We can consider the following condition* $(3)'$ *(resp.* $(4)'$*) compared with* (3) *(resp.* (4)*): in* (3) *(resp.* (4)*), replace $\mathcal{P}^*_\Phi$ with $\mathcal{P}^c_\Phi$ in the statement.*

By devissage 1.39, since $\mathcal{P}^c_\Phi \to \mathcal{G}^c_\Phi$ factors through $\mathcal{P}^*_\Phi \hookrightarrow \mathcal{G}^c_\Phi$, (4) *and* $(4)'$ *are equivalent. However,* $(3)'$ *is stronger than* (3)*. In the rest of the article, we use* (3) *instead of* $(3)'$*, mainly due to two reasons:*

(1) *The $\Delta^\circ_{\Phi,K}$-action on $\mathrm{Sh}_{K_\Phi}(P_\Phi,D_\Phi)$ fixes the $\mathcal{P}^*_\Phi$-shtuka rather than the $\mathcal{P}^c_\Phi$-shtuka,*

(2) *$P_\Phi^c \to G^c$ might not be an embedding, in practice, we need an embedding $P_\Phi^* \to G^c$ to apply the funtoriality result Proposition 4.9 (which is needed in proving the canonicity of integral models of abelian-type, see Theorem 6.26).*

In order to prove functoriality for such canonical integral models, we need further assumptions when $G^c \neq G$.

Definition 4.5. *Let $(P, \mathcal{X})$ be a Shimura datum, let Z' be a central multiplicative group in P that contains $Z(P)_{ac}$, and let $(P', \mathcal{X}') := (P, \mathcal{X})/Z'$. Let $\mathcal{P}$ be a quasi-parahoric group scheme of P, and let $\mathcal{P}'$ be a quasi-parahoric group scheme of P' defined in Definition 3.8. Let $K_p = \mathcal{P}(\mathbb{Z}_p)$ and $K'_p = \mathcal{P}'(\mathbb{Z}_p)$. Consider a system $\{\mathscr{S}_K(P, \mathcal{X})\}_{K^p}$ (resp. $\{\mathscr{S}_{K'}(P', \mathcal{X}')\}_{K',p}$) of normal flat schemes over $\mathcal{O}_E$ with generic fiber $\{\mathrm{Sh}_K(P, \mathcal{X})\}_{K^p}$ (resp. $\{\mathrm{Sh}_{K'}(P', \mathcal{X}')\}_{K',p}$), with K^p (resp. $K'^{,p}$) varying over all sufficiently small compact open subgroups of $P(\mathbb{A}_f^p)$ (resp. $P'(\mathbb{A}_f^p)$).*

*Let $\pi : (P, \mathcal{X}) \to (P', \mathcal{X}')$ be the projection. We say $\{\mathscr{S}_K(P, \mathcal{X})\}_{K^p}$ and $\{\mathscr{S}_{K'}(P', \mathcal{X}')\}_{K',p}$ are **adapted** if, for any pair $(K^p, K'^{,p})$ such that $\pi(K^p) \subset K'^{,p}$, there is a finite morphism*

$$\mathscr{S}_K(P, \mathcal{X}) \to \mathscr{S}_{K'}(P', \mathcal{X}') \tag{4.1}$$

extending the generic fiber $\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sh}_{K'}(P', \mathcal{X}')$ induced by π.

*We say $\{\mathscr{S}_K(P, \mathcal{X})\}_{K^p}$ is **adapted with respect to** $P \to P'$ if there exists $\{\mathscr{S}_{K'}(P', \mathcal{X}')\}_{K',p}$ such that $\{\mathscr{S}_K(P, \mathcal{X})\}_{K^p}$ and $\{\mathscr{S}_{K'}(P', \mathcal{X}')\}_{K',p}$ are adapted.*

Definition 4.6. *Keep notation in Definition 4.5. Let $(G, X) := (P, \mathcal{X})/W$. We say a system of normal flat models $\{\mathscr{S}_K(P, \mathcal{X})\}_{K^p}$ of $\{\mathrm{Sh}_K(P, \mathcal{X})\}_{K^p}$ adapted with respect to $P \to P'$ is moreover **adapted with respect to** $P \to G$ if there exists a system of normal flat models $\{\mathscr{S}_{K_G}(G, X)\}_{K_G^p}$ of $\{\mathrm{Sh}_{K_G}(G, X)\}_{K_G^p}$ that is adapted with $G \to G'$, where $G' = G/Z'$ (by Lemma 3.1, $Z(G)_{ac} = Z(P)_{ac} \subset Z'$) and $\mathcal{P}' = \mathcal{U} \rtimes \mathcal{G}'$, such that when K_G^p and $K_G'^{,p}$ contain the image of K^p and $K'^{,p}$ respectively, the induced morphisms on generic fibers*

$$\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sh}_{K_G}(G, X), \quad \mathrm{Sh}_{K'}(P', \mathcal{X}') \to \mathrm{Sh}_{K'_G}(G', X')$$

extend to morphisms on integral models

$$\mathscr{S}_K(P, \mathcal{X}) \to \mathscr{S}_{K_G}(G, X), \quad \mathscr{S}_{K'}(P', \mathcal{X}') \to \mathscr{S}_{K'_G}(G', X'),$$

respectively.

Recall the following fact:

Lemma 4.7. *Given a morphism between Shimura data $\iota : (P_1, \mathcal{X}_1) \to (P_2, \mathcal{X}_2)$, then $\iota(W_1) \subset W_2$ and $\iota(U_1) \subset U_2$, and ι induces $G_1 \to G_2$ and $V_1 \to V_2$. In particular, the morphism $P_1 \to P_2$ is compatible with $U_1 \rtimes G_1 \to U_2 \rtimes G_2$ in the sense of 1.33.*

Proof. Let $h_1 \in \mathcal{X}_1$, and set $h_2 = h_1 \circ \iota \in \mathcal{X}_2$. By definition, the weight filtration on $\mathrm{Lie}\, P_i$ induced by $\mathrm{Ad}_{P_i} \circ h_i$ is $W_{-1}(\mathrm{Lie}\, P_i) = \mathrm{Lie}\, W_i$, $W_{-2}(\mathrm{Lie}\, P_i) = \mathrm{Lie}\, U_i$. By construction, $\mathrm{Lie}\, P_1 \to \mathrm{Lie}\, P_2$ is a morphism between mixed Hodge structures, and it is strict; thus it maps $\mathrm{Lie}\, W_1$ to $\mathrm{Lie}\, W_2$ and $\mathrm{Lie}\, U_1$ to $\mathrm{Lie}\, U_2$. □

Lemma 4.8. *Consider one of the following two cases:*

(1) *Let $\iota : (P_1, \mathcal{X}_1) \to (P_2, \mathcal{X}_2)$ be an embedding of mixed Shimura data, let $(G_i, X_i) = (P_i, \mathcal{X}_i)/W$ be the induced pure Shimura datum. Let E_1, E_2 be the completion of reflex fields. Let $\mathcal{P}_1, \mathcal{P}_2$ be quasi-parahoric group schemes of P_1 and P_2 respectively, such that*

$$\mathcal{P}_2(\breve{\mathbb{Z}}_p) \cap P_1(\breve{\mathbb{Q}}_p) = \mathcal{P}_1(\breve{\mathbb{Z}}_p), \quad \mathcal{G}_2(\breve{\mathbb{Z}}_p) \cap G_1(\breve{\mathbb{Q}}_p) = \mathcal{G}_1(\breve{\mathbb{Z}}_p). \tag{4.2}$$

Let $K_2^p \subset P_2(\mathbb{A}_f^p)$ be a sufficiently small subgroup. One can always find $K_1^p \subset P_1(\mathbb{A}_f^p)$ such that the induced morphisms $\mathrm{Sh}_{K_1}(P_1, \mathcal{X}_1) \to \mathrm{Sh}_{K_2}(P_2, \mathcal{X}_2) \otimes_{E_2} E_1$ and $\mathrm{Sh}_{K_{G_1}}(G_1, X_1) \to$

$\mathrm{Sh}_{K_{G_2}}(G_2, X_2) \otimes_{E_2} E_1$ *are closed embeddings (such levels exist; see* [Wu25, Lem. 1.22]*), where* $K_i = \mathcal{P}_i(\mathbb{Z}_p)K_i^p$*, and* $K_{G_i} = \mathcal{G}_i(\mathbb{Z}_p)K_{G_i}^p$*, where* $K_{G_i}^p \subset G_i(\mathbb{A}_f^p)$ *are the images of* K_i^p*.*

Let $\mathcal{P}_i'$ *be a quasi-parahoric group scheme of* P_i' *as in Definition 3.8, such that* $P_1' \to P_2'$ *induces* $\mathcal{P}_1' \to \mathcal{P}_2'$*. We* assume $P_1' \to P_2'$ *is a closed embedding and moreover the smoothing* $\widetilde{\mathcal{P}}_1'$ *of the closure of* P_1' *in* $\mathcal{P}_2'$ *is a quasi-parahoric group scheme with* $(\widetilde{\mathcal{P}}_1')^\circ = \mathcal{P}_1'^\circ$*.*

(2) Let $(P, \mathcal{X}) := (P_1, \mathcal{X}_1) = (P_2, \mathcal{X}_2)$ *be a mixed Shimura datum and* $(G, X) = (G_i, X_i) = (P_i, \mathcal{X}_i)/W_i$ *be the induced pure Shimura datum, let* $E_1 = E_2$ *be the completion of the reflex field. Let* $\mathcal{P}_1 \to \mathcal{P}_2$ *be quasi-parahoric group schemes of* P *such that* $\mathcal{U}_1 = \mathcal{U}_2$ *and* $\mathcal{P}_1^\circ = \mathcal{P}_2^\circ$ *(i.e.* $\mathcal{G}_1^\circ = \mathcal{G}_2^\circ$*). Define* $\mathcal{P}_1'$*,* $\mathcal{P}_2'$ *as in Definition 3.8 such that we have a morphism* $\mathcal{P}_1' \to \mathcal{P}_2'$ *(which automatically induces* $(\mathcal{P}_1')^\circ = (\mathcal{P}_2')^\circ$ *by construction). Let* $K_i = \mathcal{P}_i(\mathbb{Z}_p)K^p$ *for some neat* $K^p \subset P(\mathbb{A}_f^p)$*, and* $K_{G_i} = \mathcal{G}_i(\mathbb{Z}_p)K_G^p$*, where* $K_G^p \subset G(\mathbb{A}_f^p)$ *are the images of* K^p*. Define* K_i'*,* K_{G_i}' *similarly, for some* $K'^{,p} \subset P'(\mathbb{A}_f^p)$*.*

Assume

$$\{\mathrm{Sh}_{K_2}(P_2, \mathcal{X}_2)\}_{K_2^p}, \quad \{\mathrm{Sh}_{K_{G_2}}(G_2, X_2)\}_{K_{G_2}^p}, \quad \{\mathrm{Sh}_{K_2'}(P_2', \mathcal{X}_2')\}_{K_2'^{,p}}, \quad \{\mathrm{Sh}_{K_{G_2}'}(G_2', X_2')\}_{K_{G_2}'^{,p}}$$

have integral models

$$\{\mathscr{S}_{K_2}(P_2, \mathcal{X}_2)\}_{K_2^p}, \quad \{\mathscr{S}_{K_{G_2}}(G_2, X_2)\}_{K_{G_2}^p}, \quad \{\mathscr{S}_{K_2'}(P_2', \mathcal{X}_2')\}_{K_2'^{,p}}, \quad \{\mathscr{S}_{K_{G_2}'}(G_2', X_2')\}_{K_{G_2}'^{,p}} \tag{4.3}$$

respectively that are normal and flat over $\mathcal{O}_{E_2}$ *and are adapted with* $P_2 \to P_2'$ *and* $P_2 \to G_2$ *in the sense of Definitions 4.5 and 4.6, such that the shtukas on generic fibers extend over these integral models, i.e., for* $(P_2, \mathcal{X}_2)$ *(and similarly for* $(P_2', \mathcal{X}_2')$*), given the commutative diagram on the right, we have horizontal morphisms on the left diagram that make the diagram commute:*

$$\begin{tikzcd}
\mathscr{S}_{K_2}(P_2, \mathcal{X}_2)^{\Diamond/} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_2', \mu', \delta=1} \arrow[d] \\
\mathscr{S}_{K_{G_2}}(G_2, X_2)^{\Diamond/} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2', \mu', \delta=1},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sh}_{K_2}(P_2, \mathcal{X}_2)^{\Diamond} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_2', \mu', \delta=1, \mathrm{Spd}\, E} \arrow[d] \\
\mathrm{Sh}_{K_{G_2}}(G_2, X_2)^{\Diamond} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2', \mu', \delta=1, \mathrm{Spd}\, E}.
\end{tikzcd}$$

Let

$$\{\mathscr{S}_{K_1}(P_1, \mathcal{X}_1)\}_{K_1^p}, \quad \{\mathscr{S}_{K_{G_1}}(G_1, X_1)\}_{K_{G_1}^p}, \quad \{\mathscr{S}_{K_1'}(P_1', \mathcal{X}_1')\}_{K_1'^{,p}}, \quad \{\mathscr{S}_{K_{G_1}'}(G_1', X_1')\}_{K_{G_1}'^{,p}}$$

be the relative normalizations of ((4.3)$\otimes \mathcal{O}_{E_1}$*) in*

$$\{\mathrm{Sh}_{K_1}(P_1, \mathcal{X}_1)\}_{K_1^p}, \quad \{\mathrm{Sh}_{K_{G_1}}(G_1, X_1)\}_{K_{G_1}^p}, \quad \{\mathrm{Sh}_{K_1'}(P_1', \mathcal{X}_1')\}_{K_1'^{,p}}, \quad \{\mathrm{Sh}_{K_{G_1}'}(G_1', X_1')\}_{K_{G_1}'^{,p}},$$

then these integral models are normal and flat over $\mathcal{O}_{E_1}$ *and are adapted with* $P_1 \to P_1'$ *and* $P_1 \to G_1$*, such that the shtukas on generic fibers extend over these integral models, i.e., for* $(P_1, \mathcal{X}_1)$ *(and similarly for* $(P_1', \mathcal{X}_1')$*), given the commutative diagram on the right, we have horizontal morphisms on the left diagram that make the diagram commute:*

$$\begin{tikzcd}
\mathscr{S}_{K_1}(P_1, \mathcal{X}_1)^{\Diamond/} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_1', \mu', \delta=1} \arrow[d] \\
\mathscr{S}_{K_{G_1}}(G_1, X_1)^{\Diamond/} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_1', \mu', \delta=1},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sh}_{K_1}(P_1, \mathcal{X}_1)^{\Diamond} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}_1', \mu', \delta=1, \mathrm{Spd}\, E} \arrow[d] \\
\mathrm{Sh}_{K_{G_1}}(G_1, X_1)^{\Diamond} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_1', \mu', \delta=1, \mathrm{Spd}\, E}.
\end{tikzcd}$$

Proof. We prove these two cases separately.

(1) Since $P_1' \to P_2'$ is a closed embedding, $G_1' \to G_2'$ is also a closed embedding. Let $\widetilde{\mathcal{G}}_1'$ be the smoothing of the closure of G_1' in $\mathcal{G}_2'$. We claim that $\widetilde{\mathcal{G}}_1'$ is a quasi-parahoric group scheme and $(\widetilde{\mathcal{G}}_1')^\circ = (\mathcal{G}_1')^\circ$: condition (4.2) shows that $\mathcal{P}_1 \to \mathcal{P}_2$ is compatible with $\mathcal{G}_1 \to \mathcal{G}_2$ by Lemma 3.6; in particular, we have compatible sections $G_1 \to P_1$ and $G_2 \to P_2$ by taking

generic fibers. Consider the induced compatible sections $G'_1 \to P'_1$ and $G'_2 \to P'_2$, $\mathcal{P}'_1 \to \mathcal{P}'_2$ is compatible with $\mathcal{G}'_1 \to \mathcal{G}'_2$ under such sections. Then the closure of G'_1 in $\mathcal{G}'_2$ is the same as the closure of G'_1 in $\mathcal{P}'_2$, hence also the closure of G'_1 in $\widetilde{\mathcal{P}}'_1$. Since $(\widetilde{\mathcal{P}}'_1)^\circ = \mathcal{P}'^\circ_1$, we have $\widetilde{\mathcal{P}}'_1 = \mathcal{U}_1 \rtimes \widetilde{\mathcal{G}}'_1$, and $(\widetilde{\mathcal{G}}'_1)^\circ = (\mathcal{G}'_1)^\circ$.

Let $\widetilde{K}'_{G_1,p} = \widetilde{\mathcal{G}}'_1(\mathbb{Z}_p)$ and $\widetilde{K}'_{G_1} = \widetilde{K}'_{G_1,p} K^p_{G_1}$. Then $\mathrm{Sh}_{\widetilde{K}'_{G_1}}(G'_1, X'_1)^\lozenge \to \mathrm{Sht}_{\widetilde{\mathcal{G}}'_1,\mu',\delta=1,\mathrm{Spd}\,E}$ extends to $\mathscr{S}_{\widetilde{K}'_{G_1}}(G'_1, X'_1)^{\lozenge/} \to \mathrm{Sht}_{\widetilde{\mathcal{G}}'_1,\mu',\delta=1}$, by [PR24, Thm. 4.3.1, 4.5.2] (with modifications in the quasi-parahoric case using [DvHKZ26, Thm. 4.1.8]). Since $(\widetilde{\mathcal{G}}'_1)^\circ = (\mathcal{G}'_1)^\circ$, we pass from $\widetilde{\mathcal{G}}'_1$ to $\mathcal{G}'_1$ using [DvHKZ26, Thm. 4.1.15]; $\mathrm{Sh}_{K'_{G_1}}(G'_1, X'_1)^\lozenge \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1,\mathrm{Spd}\,E}$ extends to $\mathscr{S}_{K'_{G_1}}(G'_1, X'_1)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1}$.

Let $P_3 = U_2 \rtimes G_1$, where G_1 acts on U_2 via the embedding $G_1 \to G_2$, and let $\mathcal{P}_3 = \mathcal{U}_2 \rtimes \mathcal{G}_1$. Moreover, let $P'_3 = U_2 \rtimes G'_1$ and $\mathcal{P}'_3 = \mathcal{U}_2 \rtimes \mathcal{G}'_1$. We can always do this since $\mathcal{P}_1 \to \mathcal{P}_2$ (resp. $\mathcal{P}'_1 \to \mathcal{P}'_2$) is compatible with $\mathcal{G}_1 \to \mathcal{G}_2$ (resp. $\mathcal{G}'_1 \to \mathcal{G}'_2$). Consider $\mathcal{P}'_3 \to \mathcal{P}'_2$ that is compatible with $\mathcal{G}'_3 = \mathcal{G}'_1 \to \mathcal{G}'_2$, by Corollary 1.49, we have a dashed morphism that makes the diagrams commute:

$$\begin{tikzcd}
\mathscr{S}_{K'_1}(P'_1, \mathcal{X}'_1)^{\lozenge/} \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & \mathscr{S}_{K'_2}(P'_2, \mathcal{X}'_2)^{\lozenge/} \arrow[d] \\
& \mathrm{Sht}_{\mathcal{P}'_3,\mu',\delta=1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sht}_{\mathcal{P}'_2,\mu',\delta=1} \arrow[d] \\
\mathscr{S}_{K'_{G_1}}(G'_1, X'_1)^{\lozenge/} \arrow[r] & \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}'_2,\mu',\delta=1}.
\end{tikzcd}$$

Apply Proposition 1.43 to the exact sequence $1 \to \mathcal{P}'_1 \to \mathcal{P}'_3 \to \mathcal{U}_2/\mathcal{U}_1 \to 1$ (the Part (2) in the proof of Proposition 1.43 does not need U_1 being normal in U_3). Then the dashed morphism $\mathscr{S}_{K'_1}(P'_1, \mathcal{X}'_1)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{P}'_3,\mu',\delta=1}$ factors through $\mathrm{Sht}_{\mathcal{P}'_1,\mu'}$, hence through $\mathrm{Sht}_{\mathcal{P}'_1,\mu',\delta=1}$; see Corollary 1.47. Compose with

$$\mathscr{S}_{K_1}(P_1, \mathcal{X}_1) \to \mathscr{S}_{K'_1}(P'_1, \mathcal{X}'_1) \quad (\textit{resp. } \mathscr{S}_{K_{G_1}}(G_1, X_1) \to \mathscr{S}_{K'_{G_1}}(G'_1, X'_1)),$$

we have

$$\mathscr{S}_{K_1}(P_1, \mathcal{X}_1)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{P}'_1,\mu',\delta=1} \quad (\textit{resp. } \mathscr{S}_{K_{G_1}}(G_1, X_1)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1})$$

extending the one on the generic fiber. The commutativity of the diagram follows from [PR24, Cor. 2.7.10].

(2) First of all, since $(\mathcal{G}'_1)^\circ = (\mathcal{G}'_2)^\circ$ by construction, $\mathrm{Sh}_{K'_{G_1}}(G', X')^\lozenge \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1,\mathrm{Spd}\,E}$ extends to $\mathscr{S}_{K'_{G_1}}(G', X')^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1}$, by [DvHKZ26, Thm. 4.1.15]. Composing with $\mathscr{S}_{K_{G_1}}(G, X) \to \mathscr{S}_{K'_{G_1}}(G', X')$, we obtain $\mathscr{S}_{K_{G_1}}(G, X)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}'_1,\mu',\delta=1}$.

By Lemma 3.1, $P' = U \rtimes G'$. By Corollary 1.49, we obtain the desired dashed morphism that extends the one on the generic fiber and makes the diagrams commute:

$$\begin{tikzcd}
\mathscr{S}_{K_1'}(P',\mathcal{X}')^{\Diamond/} \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \mathscr{S}_{K_2'}(P',\mathcal{X}')^{\Diamond/} \arrow[d] \\
& \mathrm{Sht}_{\mathcal{P}_1',\mu',\delta=1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sht}_{\mathcal{P}_2',\mu',\delta=1} \arrow[d] \\
\mathscr{S}_{K'_{G_1}}(G',X')^{\Diamond/} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_1',\mu',\delta=1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}_2',\mu',\delta=1}.
\end{tikzcd}$$

Similarly for $\mathscr{S}_{K_1}(P,\mathcal{X})^{\Diamond/} \to \mathrm{Sht}_{\mathcal{P}_1',\mu',\delta=1}$. The commutativity of the diagram follows from [PR24, Cor. 2.7.10].

□

Proposition 4.9 ([PR24, Thm. 4.3.1, 4.5.2], [DvHKZ26, Thm. 4.1.15])**.** *Consider one of the following two cases:*

(1) *Let $\iota : (G_1, X_1) \to (G_2, X_2)$ be an embedding of pure Shimura data, and let E_1, E_2 be completions of reflex fields. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be quasi-parahoric group schemes of G_1 and G_2 respectively, such that $\mathcal{G}_2(\breve{\mathbb{Z}}_p) \cap G_1(\breve{\mathbb{Q}}_p) = \mathcal{G}_1(\breve{\mathbb{Z}}_p)$. Let $K_2^p \subset G_2(\mathbb{A}_f^p)$ be any sufficiently small subgroup; one can always find $K_1^p \subset G_1(\mathbb{A}_f^p)$ such that the induced morphism $\mathrm{Sh}_{K_1}(G_1, X_1) \to \mathrm{Sh}_{K_2}(G_2, X_2)$ is a closed embedding, where $K_i = \mathcal{G}_i(\mathbb{Z}_p)K_i^p$. Let $[\Phi_1] = [(Q_{\Phi_1}, X^+_{\Phi_1}, g_{\Phi_1})] \in \mathrm{Cusp}_{K_1}(G_1, X_1)$, and $[\Phi_2] = [\iota_*\Phi_1] = [(Q_{\Phi_2}, X^+_{\Phi_2}, g_{\Phi_2})] \in \mathrm{Cusp}_{K_2}(G_2, X_2)$ (with $g_{\Phi_1} = g_{\Phi_2}$). Here we can vary K_1^p (resp. K_2^p), and do not distinguish Φ_1 (resp. Φ_2) for different levels K_1^p (resp. K_2^p) once we have prescribed $g = g_{\Phi_1} = g_{\Phi_2} \in G(\mathbb{A}_f)$. Assume*
   (a) *$G_1^c \to G_2^c$ is an embedding, and the smoothing $\widetilde{\mathcal{G}}_1^c$ of the closure of G_1^c in $\mathcal{G}_2^c$ is a quasi-parahoric group scheme with $(\widetilde{\mathcal{G}}_1^c)^\circ = (\mathcal{G}_1^c)^\circ$.*
   (b) *$\mathcal{P}_{\Phi_2}(\breve{\mathbb{Z}}_p) \cap P_{\Phi_1}(\breve{\mathbb{Q}}_p) = \mathcal{P}_{\Phi_1}(\breve{\mathbb{Z}}_p)$ induces $\mathcal{G}_{\Phi_2,h}(\breve{\mathbb{Z}}_p) \cap G_{\Phi_1,h}(\breve{\mathbb{Q}}_p) = \mathcal{G}_{\Phi_1,h}(\breve{\mathbb{Z}}_p)$.*

(2) *Let $(G, X) := (G_1, X_1) = (G_2, X_2)$ be a pure Shimura datum, and let $E_1 = E_2$ be the completion of the reflex field. Let $\mathcal{G}_1 \to \mathcal{G}_2$ be quasi-parahoric group schemes of G such that $\mathcal{G}_1^\circ = \mathcal{G}_2^\circ$. Fix $(Q_\Phi, X^+_\Phi, g_\Phi) := (Q_{\Phi_1}, X^+_{\Phi_1}, g_{\Phi_1}) = (Q_{\Phi_2}, X^+_{\Phi_2}, g_{\Phi_2})$ as above. Let $K_i = \mathcal{G}_i(\mathbb{Z}_p)K^p$ for some neat $K^p \subset G(\mathbb{A}_f^p)$.*

Assume $\{\mathrm{Sh}_{K_2}\}_{K_2^p}$ and $\{\mathrm{Sh}_{K_2^c}\}_{K_2^{c,p}}$ have canonical integral models $\{\mathscr{S}_{K_2}\}_{K_2^p}$ and $\{\mathscr{S}_{K_2^c}\}_{K_2^{c,p}}$ respectively that are adapted with $G_2 \to G_2^c$. Let $\{\mathscr{S}_{K_1}\}_{K_1^p}$ and $\{\mathscr{S}_{K_1^c}\}_{K_1^{c,p}}$ be the relative normalizations of $\{\mathscr{S}_{K_2}\}_{K_2^p}$ and $\{\mathscr{S}_{K_2^c}\}_{K_2^{c,p}}$ in $\{\mathrm{Sh}_{K_1}\}_{K_1^p}$ and $\{\mathrm{Sh}_{K_1^c}\}_{K_1^{c,p}}$ respectively. In case (2), we ***further assume*** *$\{\mathscr{S}_{K_1} \to \mathscr{S}_{K_1^c}\}_{K_1^p}$* ***are étale****. Then $\{\mathscr{S}_{K_1}\}_{K_1^p}$ and $\{\mathscr{S}_{K_1^c}\}_{K_1^{c,p}}$ are canonical integral models of $\{\mathrm{Sh}_{K_1}\}_{K_1^p}$ and $\{\mathrm{Sh}_{K_1^c}\}_{K_1^{c,p}}$ respectively, adapted with $G_1 \to G_1^c$.*

Moreover, assume

$$\{\mathrm{Sh}_{K_{\Phi_2}}\}_{K^p_{\Phi_2}}, \quad \{\mathrm{Sh}_{K^*_{\Phi_2}}\}_{K^{*,p}_{\Phi_2}}, \quad \{\mathrm{Sh}_{K_{\Phi_2,h}}\}_{K^p_{\Phi_2,h}}, \quad \{\mathrm{Sh}_{K^*_{\Phi_2,h}}\}_{K^{*,p}_{\Phi_2,h}} \tag{4.4}$$

have canonical integral models

$$\{\mathscr{S}_{K_{\Phi_2}}\}_{K^p_{\Phi_2}}, \quad \{\mathscr{S}_{K^*_{\Phi_2}}\}_{K^{*,p}_{\Phi_2}}, \quad \{\mathscr{S}_{K_{\Phi_2,h}}\}_{K^p_{\Phi_2,h}}, \quad \{\mathscr{S}_{K^*_{\Phi_2,h}}\}_{K^{*,p}_{\Phi_2,h}} \tag{4.5}$$

respectively, adapted with $P_{\Phi_2} \to P^*_{\Phi_2}$ and $P_{\Phi_2} \to G_{\Phi_2,h}$. Let

$$\{\mathscr{S}_{K_{\Phi_1}}\}_{K^p_{\Phi_1}}, \quad \{\mathscr{S}_{K^*_{\Phi_1}}\}_{K^{*,p}_{\Phi_1}}, \quad \{\mathscr{S}_{K_{\Phi_1,h}}\}_{K^p_{\Phi_1,h}}, \quad \{\mathscr{S}_{K^*_{\Phi_1,h}}\}_{K^{*,p}_{\Phi_1,h}} \tag{4.6}$$

be the relative normalizations of (4.5) in

$$\{\mathrm{Sh}_{K_{\Phi_1}}\}_{K^p_{\Phi_1}}, \quad \{\mathrm{Sh}_{K^*_{\Phi_1}}\}_{K^{*,p}_{\Phi_1}}, \quad \{\mathrm{Sh}_{K_{\Phi_1,h}}\}_{K^p_{\Phi_1,h}}, \quad \{\mathrm{Sh}_{K^*_{\Phi_1,h}}\}_{K^{*,p}_{\Phi_1,h}} \tag{4.7}$$

respectively. In case (2), *we* ***further assume*** $\{\mathscr{S}_{K_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_1}}\}_{K^p_{\Phi_1}}$ ***and*** $\{\mathscr{S}_{K_{\Phi_1,h}} \to \mathscr{S}_{K^*_{\Phi_1,h}}\}_{K^p_{\Phi_1,h}}$ ***are étale****. Then (4.6) are canonical integral models of (4.7) respectively, adapted with* $P_{\Phi_1} \to P^*_{\Phi_1}$ *and* $P_{\Phi_1} \to G_{\Phi_1,h}$*.*

Proof. Axiom (1) is automatic; it comes from the construction of the relative normalization, see the first paragraph in the proof of [DvHKZ26, Thm. 4.1.15]. Axiom (2) can be verified using axiom (4), since $\lozenge$ is full-faithful on the category of flat and normal formal schemes locally formally of finite type over $\mathcal{O}_E$; see [SW20, Prop. 18.4.1]. We only need to consider axioms (3) and (4).

For (G_1, X_1):

(1) We first verify that $\{\mathscr{S}_{K_1^c}(G_1^c, X_1^c)\}_{K_1^{c,p}}$ is canonical in the sense of the Pappas–Rapoport conjectural framework; this is [PR24, Thm. 4.3.1, 4.5.2] (with quasi-parahoric modifications using [DvHKZ26, Thm. 4.1.8], and with modifications in the $G \neq G^c$ case using the second paragraph of the proof of Lemma 4.8, case (1)). Next, we show $\{\mathscr{S}_{K_1}(G_1, X_1)\}_{K_1^p}$ is canonical. The extension of shtukas is given by composing $\mathscr{S}_{K_1} \to \mathscr{S}_{K_1^c}$ with $\mathscr{S}_{K_1^c} \to \mathrm{Sht}_{\mathcal{G}_1^c,\mu_1^c,\delta=1}$. Let $x \in \mathscr{S}_{K_1}(k)$ and $\bar{x} \in \mathscr{S}_{K_1^c}(k)$ be its image. The existence of an isomorphism $\Theta_x : ((\mathscr{S}_{K_1})^\wedge_{/x})^\lozenge \cong (\mathcal{M}^{\mathrm{int}}_{\mathcal{G}_1^c,b_x,\mu_1^c})^\wedge_{/x_0}$ follows from the same arguments as in [PR24, §4.7, 4.8], and Θ_x is compatible with $\Theta_{\bar{x}}$ by construction. This in turn shows that the finite morphism $\mathscr{S}_{K_1} \to \mathscr{S}_{K_1^c}$ is étale, and $\{\mathscr{S}_{K_1}\}_{K_1^p}$ is adapted with $G_1 \to G_1^c$.

(2) The canonicity of $\{\mathscr{S}_{K_1^c}(G_1^c, X_1^c)\}_{K_1^{c,p}}$ comes from [DvHKZ26, Thm. 4.1.15]. Let us show the canonicity of $\{\mathscr{S}_{K_1}(G_1, X_1)\}_{K_1^p}$. Note that $\mathscr{S}_{K_1} \to \mathscr{S}_{K_2}$ is finite étale since $\mathscr{S}_{K_1} \to \mathscr{S}_{K_1^c}$ and $\mathscr{S}_{K_2} \to \mathscr{S}_{K_2^c}$ are finite étale by assumption, and $\mathscr{S}_{K_1^c} \to \mathscr{S}_{K_2^c}$ is finite étale by axiom (4); this implies axiom (4). The extension of shtukas comes from Lemma 4.8, case (2).

For (P_{Φ_1}, D_{Φ_1}) and $(G_{\Phi_1,h}, D_{\Phi_1,h})$, apply Lemma 4.8, and let $P_i' = P^*_{\Phi_i}$, $\mathcal{P}_i' = \mathcal{P}^*_{\Phi_i}$. Verify the conditions in Lemma 4.8:

(1) We need to show that $P_1' \to P_2'$ is a closed embedding and that the smoothing $\widetilde{\mathcal{P}}_1'$ of the closure of P_1' in $\mathcal{P}_2'$ is a quasi-parahoric group scheme with $(\widetilde{\mathcal{P}}_1')^\circ = (\mathcal{P}_1')^\circ$. Since $G_1^c \hookrightarrow G_2^c$ and $P_i' \hookrightarrow G_i^c$, we have $P_1' \hookrightarrow P_2'$. We have

$$\widetilde{\mathcal{P}}_1'(\breve{\mathbb{Z}}_p) = P_1'(\breve{\mathbb{Q}}_p) \cap \mathcal{P}_2'(\breve{\mathbb{Z}}_p) = P_1'(\breve{\mathbb{Q}}_p) \cap \mathcal{G}_2^c(\breve{\mathbb{Z}}_p) = P_1'(\breve{\mathbb{Q}}_p) \cap \mathcal{G}_1^c(\breve{\mathbb{Z}}_p) = \mathcal{P}_1'(\breve{\mathbb{Z}}_p).$$

Since a smooth affine group scheme of a given group is uniquely determined by its set of $\breve{\mathbb{Z}}_p$-points, we have $\widetilde{\mathcal{P}}_1' = \mathcal{P}_1'$.

(2) We need to show that $\mathcal{G}_1 \to \mathcal{G}_2$ induces $\mathcal{P}_{\Phi_1} \to \mathcal{P}_{\Phi_2}$ such that $\mathcal{P}^\circ_{\Phi_1} = \mathcal{P}^\circ_{\Phi_2}$. This is standard, since $(\mathcal{P}_{\Phi_i}, \mu_{\Phi_i})$ comes from the boundary in the sense of Definition 1.21; see Lemma 3.3. On the other hand, $\mathcal{G}_1^c \to \mathcal{G}_2^c$ induces $\mathcal{P}_1' \to \mathcal{P}_2'$ such that $(\mathcal{P}_1')^\circ = (\mathcal{P}_2')^\circ$.

Now let us show canonicity of integral models. For $(G_{\Phi_1,h}, D_{\Phi_1,h})$:

(1) This is similar to (G_1, X_1), but we need to handle the case where $G_{\Phi_1,h} \to G^*_{\Phi_1,h}$ has non-connected kernel. By assumption, the smoothing $\widetilde{\mathcal{G}}^*_{\Phi_1,h}$ of the closure of $G^*_{\Phi_1,h}$ in $\mathcal{G}^*_{\Phi_2,h}$ is a quasi-parahoric group scheme and $(\widetilde{\mathcal{G}}^*_{\Phi_1,h})^\circ = (\mathcal{G}^*_{\Phi_1,h})^\circ$; see the first paragraph of the proof of Lemma 4.8. For the extension of shtukas, we apply Lemma 4.8, case (1). For axiom (4), we apply results (1) and (2) successively from the first two paragraphs of this proof.

(2) This is similar to (G_1, X_1).

For (P_{Φ_1}, D_{Φ_1}):

(1) For extension of shtukas, we apply Lemma 4.8 Case (1). Note that we can adjust level away from p such that $\mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$, $\mathrm{Sh}_{K_{\Phi_1}} \to \mathrm{Sh}_{K_{\Phi_2}}$ and $\mathrm{Sh}_{K_{\Phi_1,h}} \to \mathrm{Sh}_{K_{\Phi_2,h}}$ are all closed

embeddings by [Wu25, Lem. 1.22]. Axiom (4) follows from the same arguments as in [PR24, §4.7, 4.8]; we only explain where the arguments need to be adjusted or verified in our case. We use notation from [PR24, §4.7, 4.8]. The base point x_0 in $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_{\Phi_2},b_{\Phi_2,x},\mu^c_{\Phi_2}}$ is the image of the base point in $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_{\Phi_1},b_{\Phi_1,x},\mu^c_{\Phi_1}}$ (with $b_{\Phi_2,x} = b_{\Phi_1,x}$). The arguments in [PR24, §4.7.2] (with modifications in [PR24, §4.8]) imply that there is a $\mathrm{Spd}\,\hat{R}_x$-point (where $\mathrm{Spf}\,\hat{R}_x = \mathscr{S}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1})^{\wedge}_{/x}$) of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_{\Phi_1},b_{\Phi_1,x},\mu^c_{\Phi_1}}$ lifting the base point x_0, such that the corresponding $\mathcal{G}^c_{\Phi_1}$-shtuka is equal to the pushout $\mathcal{G}^c_{\Phi_1} \times^{\mathcal{P}^*_{\Phi_1}}$ of the $\mathcal{P}^*_{\Phi_1}$-shtuka coming from $\mathscr{P}_{K_{\Phi_1}}$. In the arguments below [PR24, Prop. 4.7.1], note that $\mathrm{Spd}(\hat{R}_x)_\eta$ and $(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c_{\Phi_1},b_{\Phi_1,x},\mu^c_{\Phi_1}})^{\wedge}_{/x_0}$ have the same dimension (see Remark 1.16). Similarly for $\mathscr{S}_{K^*_{\Phi_1}}(P^*_{\Phi_1}, D^*_{\Phi_1})$, and $\Theta_{\bar{x}}$ is compatible with Θ_x, where $\bar{x} \in \mathscr{S}_{K^*_{\Phi_1}}(k)$ is the projection of x. This in particular shows that the finite morphism $\mathscr{S}_{K_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_1}}$ is étale, and $\{\mathscr{S}_{K_{\Phi_1}}\}_{K^p_{\Phi_1}}$ is adapted with $P_{\Phi_1} \to P^*_{\Phi_1}$.

(2) For the extension of shtukas, we apply Lemma 4.8, case (2). Axiom (4) for $\mathscr{S}_{K^*_{\Phi_1}}$: we need to show $\mathscr{S}_{K^*_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_2}}$ is étale, and this follows from the same process as the proof of [DvHKZ26, Thm. 4.1.15]: Note that $\mathrm{Sh}_{K^*_{\Phi_1}} \to \mathrm{Sh}_{K^*_{\Phi_2}}$ is a finite étale torsor under the finite abelian group $K^*_{\Phi_2,p}/K^*_{\Phi_1,p} = \pi_0(\mathcal{P}^*_{\Phi_2})^{\phi}/\pi_0(\mathcal{P}^*_{\Phi_1})^{\phi}$ since $(P^*_i)^c = P^*_i$, and by [DvHKZ26, Prop. 2.3.1] and Corollary 1.49, we have a cartesian diagram:

$$\begin{tikzcd}
\mathscr{S}_{K^*_{\Phi_1}}(P^*_{\Phi_1}, D^*_{\Phi_1})^{\Diamond/} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{P}^*_{\Phi_1},\mu^*_1,\delta=1} \arrow[d] \\
\mathscr{S}_{K^*_{\Phi_2}}(P^*_{\Phi_2}, D^*_{\Phi_2})^{\Diamond/} \arrow[r] & \mathrm{Sht}_{\mathcal{P}^*_{\Phi_2},\mu^*_2,\delta=1},
\end{tikzcd}$$

where $\mathrm{Sht}_{\mathcal{P}^*_{\Phi_1},\mu^*_1,\delta=1,E} \to \mathrm{Sht}_{\mathcal{P}^*_{\Phi_2},\mu^*_2,\delta=1,E}$ is a finite étale torsor under $\pi_0(\mathcal{P}^*_{\Phi_2})^{\phi}/\pi_0(\mathcal{P}^*_{\Phi_1})^{\phi}$, and $\mathscr{S}_{K^*_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_2}}$ defined via relative normalization is also a a finite étale torsor under $\pi_0(\mathcal{P}^*_{\Phi_2})^{\phi}/\pi_0(\mathcal{P}^*_{\Phi_1})^{\phi}$. Axiom (4) for $\mathscr{S}_{K_{\Phi_1}}$ then follows from the étaleness of $\mathscr{S}_{K_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_1}}$ (which also implies $\mathscr{S}_{K_{\Phi_1}} \to \mathscr{S}_{K_{\Phi_2}}$ is étale since $\mathscr{S}_{K_{\Phi_2}} \to \mathscr{S}_{K^*_{\Phi_2}}$ and $\mathscr{S}_{K^*_{\Phi_1}} \to \mathscr{S}_{K^*_{\Phi_2}}$ are étale).

□

Proposition 4.10 ([PR24, Thm. 4.2.4] and [DvHKZ26, Prop. 4.1.10])**.** *Let $\iota : (G_1, X_1) \to (G_2, X_2)$ be a morphism of Shimura data (not necessarily an embedding), and let E_1 and E_2 be completions of reflex fields. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be quasi-parahoric group schemes of G_1 and G_2 respectively, such that $G_1 \to G_2$ induces $\mathcal{G}_1 \to \mathcal{G}_2$. Let $[\Phi_1] = [(Q_{\Phi_1}, X^+_{\Phi_1}, g_{\Phi_1})] \in \mathrm{Cusp}_{K_1}(G_1, X_1)$, and $[\Phi_2] = [\iota_*\Phi_1] = [(Q_{\Phi_2}, X^+_{\Phi_2}, g_{\Phi_2})] \in \mathrm{Cusp}_{K_2}(G_2, X_2)$. Assume $\{\mathrm{Sh}_{K_{\Phi_i}}(P_{\Phi_i}, D_{\Phi_i})\}_{K^p_i}$ have canonical models $\{\mathscr{S}_{K_{\Phi_i}}(P_{\Phi_i}, D_{\Phi_i})\}_{K^p_{\Phi_i}}$ that are adapted with $P_{\Phi_i} \to P^*_{\Phi_i}$ and $P_{\Phi_i} \to G_{\Phi_i,h}$ for $i = 1, 2$. Then the induced morphism $\mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \to \mathrm{Sh}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}) \otimes_{E_1} E_2$ extends canonically to*

$$\mathscr{S}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \to \mathscr{S}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}) \otimes_{\mathcal{O}_{E_1}} \mathcal{O}_{E_2}.$$

Proof. Consider the diagonal embedding of Shimura data $(G_1, X_1) \hookrightarrow (G_1 \times G_2, X_1 \times X_2)$. Then $[\Phi_1] = [(Q_{\Phi_1}, X^+_{\Phi_1}, g_{\Phi_1})] \in \mathrm{Cusp}_{K_1}(G_1, X_1)$ induces $[\Phi_1 \times \Phi_2] = [(Q_{\Phi_1} \times Q_{\Phi_2}, X^+_{\Phi_1} \times X^+_{\Phi_2}, g_{\Phi_1} \times g_{\Phi_2})] \in \mathrm{Cusp}_{K_1 \times K_2}(G_1 \times G_2, X_1 \times X_2)$. Note that $\mathcal{G}_1(\breve{\mathbb{Z}}_p) = G_1(\breve{\mathbb{Q}}_p) \cap (\mathcal{G}_1(\breve{\mathbb{Z}}_p) \times \mathcal{G}_2(\breve{\mathbb{Z}}_p))$, $(G_1 \times G_2)^c = G^c_1 \times G^c_2$, the closure of G^c_1 in $\mathcal{G}^c_1 \times \mathcal{G}^c_2$ is exactly $\mathcal{G}^c_1$. Moreover, $\mathcal{G}_{\Phi_1,h}(\breve{\mathbb{Z}}_p) = G_{\Phi_1,h}(\breve{\mathbb{Q}}_p) \cap (\mathcal{G}_{\Phi_1,h}(\breve{\mathbb{Z}}_p) \times \mathcal{G}_{\Phi_2,h}(\breve{\mathbb{Z}}_p))$. Then we apply Proposition 4.9 case (1), and get a canonical model $\{\mathscr{S}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1})'\}_{K^p_{\Phi_1}}$ of $\{\mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1})\}_{K^p_{\Phi_1}}$. We need to show that the projection to the first factor $\mathscr{S}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1})' \to \mathscr{S}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1})$ is an isomorphism. We apply arguments from the proof of [DvHKZ26, Prop. 4.1.10], and use the pushout $\mathcal{G}^c_{\Phi_i}$-shtukas instead of $\mathcal{P}^*_{\Phi_i}$-shtukas. ☐

Corollary 4.11 ([PR24, Thm. 4.2.4] and [DvHKZ26, Cor. 4.1.13])**.** *A canonical integral model* $\{\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)\}_{K^p_\Phi}$ *of* $\{\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)\}_{K^p_\Phi}$ *that is adapted with* $P_\Phi \to P^*_\Phi$ *and* $P_\Phi \to G_{\Phi,h}$ *is unique up to a unique isomorphism, if it exists.*

Proof. This follows from Proposition 4.10 (see the proof of [DvHKZ26, Cor. 4.1.13]). ☐

5. CANONICAL EXTENSIONS ON TOROIDAL COMPACTIFICATIONS

5.1. **Axiomatic setup of good compactifications.** Let (G, X) be a Shimura datum. Fix an open compact subgroup $K \subset G(\mathbb{A}_f)$, and assume that $K = K_pK^p$, where $K^p \subset G(\mathbb{A}^p_f)$ is neat open compact and K_p is open compact. Choose an admissible (rational polyhedral) smooth projective cone decomposition Σ (without self-intersections). Denote the toroidal compactification by $\mathrm{Sh}^\Sigma_K := \mathrm{Sh}^\Sigma_K(G, X)$ and the minimal compactification by $\mathrm{Sh}^{\min}_K := \mathrm{Sh}^{\min}_K(G, X)$; they are defined over the reflex field $\mathbb{E} := \mathbb{E}(G, X)$. There is a proper morphism from Sh^Σ_K to $\mathrm{Sh}^{\min}_K$ that is compatible with the stratifications on the source and the target.

For integral models, a similar story is expected. The properties of these integral models are summarized as [LS18a, Prop. 2.1.2] and [MP19, Thm. 4.1.5]:

Axiom 5.1 (Qualitative descriptions of good compactifications; [LS18b, Prop. 2.2] and [MP19, Thm. 4.1.5 and Thm. 5.2.11])**.** *Fix a prime number p and a place v of $\mathbb{E}$ over p. Set $E := \mathbb{E}_v$. For the cone decomposition Σ above*[3]*, there is a normal, proper, flat model $\mathscr{S}^\Sigma_K$ for $\mathrm{Sh}^\Sigma_{K,E}$ over $\mathcal{O}_E$, and also a normal, projective, flat model $\mathscr{S}^{\min}_K$ for $\mathrm{Sh}^{\min}_{K,E}$ over $\mathcal{O}_E$. The following properties hold for $\mathscr{S}^\Sigma_K$ and $\mathscr{S}^{\min}_K$:*

(5.1.1) *There is a proper surjective morphism $\oint^\Sigma_K : \mathscr{S}^\Sigma_K \to \mathscr{S}^{\min}_K$ with geometrically connected fibers. The map $\oint^\Sigma_K$ extends the one constructed in the characteristic zero theory. There are open dense embeddings $J^\Sigma : \mathscr{S}_K \to \mathscr{S}^\Sigma_K$ and $J^{\min} : \mathscr{S}_K \to \mathscr{S}^{\min}_K$ such that $\oint^\Sigma_K \circ J^\Sigma = J^{\min}$.*

(5.1.2) *There is a stratification of locally closed subschemes for $\mathscr{S}^\Sigma_K$,*

$$\mathscr{S}^\Sigma_K := \coprod_{\Upsilon := [(\Phi,\sigma)] \in \mathrm{Cusp}_K(G,X,\Sigma)} \mathcal{Z}_{[(\Phi,\sigma)],K},$$

and a stratification of locally closed subschemes for $\mathscr{S}^{\min}_K$,

$$\mathscr{S}^{\min}_K := \coprod_{[\Phi] \in \mathrm{Cusp}_K(G,X)} \mathcal{Z}_{[\Phi],K}$$

extending the stratifications for Sh^Σ_K and $\mathrm{Sh}^{\min}_K$, respectively. Each stratum is normal and is flat over $\mathcal{O}_E$. The same partial orders among strata and the same expressions of closures of strata as the characteristic zero theory hold for $\{\mathcal{Z}_{[(\Phi,\sigma)],K}\}$ and $\{\mathcal{Z}_{[\Phi],K}\}$. The preimage $\oint^\Sigma_K(\mathcal{Z}_{[\Phi],K})$ is the union of the strata labeled by preimage of $[\Phi]$ in $\mathrm{Cusp}_K(G, X, \Sigma)$ under the projection $\mathrm{Cusp}_K(G, X, \Sigma) \to \mathrm{Cusp}_K(G, X)$. For the definitions of $\mathrm{Cusp}_K(G, X)$ and $\mathrm{Cusp}_K(G, X, \Sigma)$, see, e.g., [Wu25, §1.1 and §1.3].

For any cusp label representative Φ, there is a cone decomposition $\Sigma^+(\Phi)$ associated, which decomposes an open self-adjoint nondegenerated cone $\mathbf{P}^+_\Phi$; the σ's in $[(\Phi, \sigma)]$ are taken from $\Sigma^+(\Phi)$.

---

[3] In practice, it is usually harmless to refine the cone decompositions, as long as the refinements are still smooth and projective or are refinements of some smooth projective cone decompositions.

(5.1.3) *There is a partial-ordered set* $\mathcal{CLR}(G,X)$ *of cusp label representatives from characteristic zero theory; the quotient of* $\mathcal{CLR}(G,X)$ *by the equivalence relations induces* $\mathrm{Cusp}_K(G,X)$*. For each* $\Phi\in\mathcal{CLR}(G,X)$*, one can associate a mixed Shimura datum* (P_Φ,D_Φ) *in the sense of* [Pin90, Def. 2.1]*. The mixed Shimura variety* $\mathrm{Sh}_{K_\Phi}:=\mathrm{Sh}_{K_\Phi}(P_\Phi,D_\Phi)$ *admits a tower*

$$\mathrm{Sh}_{K_\Phi}(P_\Phi,D_\Phi)\to\mathrm{Sh}_{\overline{K}_\Phi}(\overline{P}_\Phi,\overline{D}_\Phi)\to\mathrm{Sh}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h}),$$

where the first map is a torsor under a split torus $\mathbf{E}_{K_\Phi}$ *and the second map is a torsor under an abelian scheme over* $\mathrm{Sh}_{K_{\Phi,h}}$*. The variety* $\mathrm{Sh}_{K_{\Phi,h}}$ *is a pure Shimura variety over* E*.*

There is a tower

$$\mathscr{S}_{K_\Phi}\to\mathscr{S}_{\overline{K}_\Phi}\to\mathscr{S}_{K_{\Phi,h}} \tag{5.1}$$

of normal flat schemes of finite type over $\mathcal{O}_E$*, such that the tower extends the one displayed above. The first map is also a torsor under the same split torus. The second map is proper and surjective.*

For any equivalence $\Phi\xrightarrow{\sim}\Phi'$ *in* $\mathcal{CLR}(G,X)$*, the corresponding towers of integral models as in* (5.1) *are canonically isomorphic.*

(5.1.4) *There is a commutative diagram*

$$\begin{tikzcd}
\mathscr{S}_K^{\Sigma}\supset\mathcal{Z}_{[(\Phi,\sigma)],K} \arrow[dd] & \mathscr{S}_{K_\Phi,\sigma} \arrow[l, "/\Delta^\circ_{\Phi,K}"'] \arrow[r, hook] & \mathscr{S}_{K_\Phi}(\sigma) \arrow[d] \\
 & & \overline{\mathscr{S}}_{K_\Phi} \arrow[d] \\
\mathscr{S}_K^{\min}\supset\mathcal{Z}_{[\Phi],K} & & \mathscr{S}_{K_{\Phi,h}}. \arrow[ll, "/\Delta_{\Phi,K}"']
\end{tikzcd} \tag{5.2}$$

In the diagram above, there are groups $\Delta^\circ_{\Phi,K}\triangleleft\Delta_{\Phi,K}$ *acting on* $\mathscr{S}_{K_\Phi}(\sigma)$ *and* $\mathscr{S}_{K_{\Phi,h}}$*, respectively. Their actions factor through finite quotients, and the quotient schemes have canonical isomorphisms* $\mathcal{Z}_{[(\Phi,\sigma)],K}\cong\Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi,\sigma}$ *and* $\mathcal{Z}_{[\Phi],K}\cong\Delta_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}$*. The scheme* $\mathscr{S}_{K_\Phi,\sigma}$ *is the* σ*-stratum of the (relative) affine toric embedding* $\mathscr{S}_{K_\Phi}\hookrightarrow\mathscr{S}_{K_\Phi}(\sigma)$ *with respect to the cone* σ*.*

For the definitions of $\Delta^\circ_{\Phi,K}$ *and* $\Delta_{\Phi,K}$*, see* [Wu25, §1.3]*. In fact, there is an equivariant action of* $\Delta_{\Phi,K}$ *on* (5.1)*, and* $\Delta^\circ_{\Phi,K}$ *is the normal subgroup fixing the induced action on the cone decomposition (which is independent of the choice of cones).*

Moreover, the quotient $\mathscr{S}^*_{K_\Phi}:=\Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}\to\overline{\mathscr{S}}^*_{K_\Phi}:=\Delta^\circ_{\Phi,K}\backslash\overline{\mathscr{S}}_{K_\Phi}$ *is also a torsor under a split torus* $\mathbf{E}_{\widetilde{K}_\Phi}$*.*

(5.1.5) *There is a strata-preserving isomorphism*

$$\mathfrak{X}_{\Upsilon,K}:=\mathfrak{X}_{[(\Phi,\sigma)],K}:=(\mathscr{S}_K^{\Sigma})^{\wedge}_{\mathcal{Z}_{[(\Phi,\sigma)],K}}\cong\Delta^\circ_{\Phi,K}\backslash(\mathscr{S}_{K_\Phi}(\sigma))^{\wedge}_{\mathscr{S}_{K_\Phi,\sigma}}.$$

(5.1.6) *There is a stronger strata-preserving isomorphism*

$$\mathfrak{X}^\circ_{\Upsilon,K}\cong\Delta^\circ_{\Phi,K}\backslash(\mathscr{S}_{K_\Phi}(\sigma))^{\wedge}_{\mathscr{S}^+_{K_\Phi,\sigma}}.$$

The scheme $\mathscr{S}^+_{K_\Phi,\sigma}$ *denotes* $\cup_{\tau\subset\sigma,\tau\in\Sigma^+(\Phi)}\mathscr{S}_{K_\Phi,\tau}$ *and* $\mathfrak{X}^\circ_{\Upsilon,K}:=(\mathscr{S}_K^{\Sigma})^{\wedge}_{\mathcal{Z}^+_{\Upsilon,K}}$*, where* $\mathcal{Z}^+_{\Upsilon,K}:=\cup_{\tau\subset\sigma,\tau\in\Sigma^+(\Phi)}\mathcal{Z}_{[(\Phi,\tau)],K}$*.*

Convention 5.2. *Denote* $\mathscr{S}^{+,*}_{K_\Phi,\sigma}:=\Delta^\circ_{\Phi,K}\backslash\mathscr{S}^+_{K_\Phi,\sigma}$*. When* Φ *and* K *are clear, we denote* $\mathfrak{X}_{\Upsilon,K}$*,* $\mathfrak{X}^\circ_{\Upsilon,K}$*,* $\mathscr{S}^{+,(*)}_{K_\Phi,\sigma}$ *and* $\mathcal{Z}^+_{\Upsilon,K}$ *by* $\mathfrak{X}_\sigma$*,* $\mathfrak{X}^\circ_\sigma$*,* $\mathscr{S}^{+,(*)}_\sigma$ *and* $\mathcal{Z}^+_\sigma$*, respectively.*

Over the generic fiber, good compactifications satisfying the above axioms was proved by Pink [Pin90] *based on* [AMRT10]. *The notation system for the generic fiber can be obtained by replacing* $\mathscr{S}$ *with* Sh *and replacing* $\mathscr{Z}$ *with* Z.

Remark 5.3. *The main difference between Axiom 5.1 and* [LS18b, Prop. 2.2] *is that we need to treat the case where* $\Delta^{\circ}_{\Phi,K}$ *is nontrivial. The item (9) in* loc. cit. *is in fact a consequence of having a strata-preserving isomorphism between completions as in (5.1.5) (see, e.g.,* [Wu25, Prop. 4.53]*).*

Axiom (5.1.6) was proved in the Hodge-type case in [LS18a, Prop. 2.1.3]. In fact, Axiom (5.1.6) is also true in the abelian-type case. The proof is the same as [Wu25, Prop. 4.32 and Lem. 4.49] with only some symbols changed; let us record it below.

Proposition 5.4. *Axiom (5.1.6) is true for integral models* $\mathscr{S}^{\Sigma'_2}_{K_2}$ *stated in* [Wu25, Thm. 4.39].

Proof. In this proof, we will use some conventions in [Wu25] without explanation; the readers are recommended to consult the list of symbols there.

By replacing "$\mathcal{S}_{\widetilde{K}_{\Phi},\sigma}$" with $\mathscr{S}^{+}_{\widetilde{K}_{\Phi},\sigma} := \cup_{\tau\subset\sigma,\tau\in\Sigma^{+}(\Phi)}\mathscr{S}_{\widetilde{K}_{\Phi},\tau}$ and replacing "$\mathscr{Z}_{[ZP^{b}(\Phi,\sigma)],K}$" with $\mathscr{Z}^{+}_{[ZP^{b}(\Phi,\sigma)],K} := \cup_{\tau\subset\sigma,\tau\in\Sigma^{+}(\Phi)}\mathscr{Z}_{[ZP^{b}(\Phi,\tau)],K}$, the argument in [Wu25, Prop. 4.32] goes through verbatim with the following exception: We need to see that $[\sigma]_{ZP}$ and $[\tau]_{ZP}$ are of the same cardinality. But $[\tau]_{ZP}$ is defined to be the Δ-orbit in $\Delta_{ZP,K}$-orbit of τ, where $\Delta\subset\Delta^{ZP}_{\Phi,K}$ is some subgroup independent of the choice of τ. Also, the stabilizer $\Delta^{ZP,\circ}_{\Phi,K}$ of the $\Delta^{ZP}_{\Phi,K}$-action is independent of the choice of τ. We now have

$$(\mathscr{S}^{\Sigma}_{K,\mathcal{O}_{K_Z}})^{\wedge}_{\mathscr{Z}^{+}_{[ZP^{b}(\Phi,\sigma)],K,\mathcal{O}_{K_Z}}} \cong (\mathscr{S}_{\widetilde{K}_{\Phi}}(\sigma)_{\mathcal{O}_{K_Z}})^{\wedge}_{\mathscr{S}^{+}_{\widetilde{K}_{\Phi},\sigma,\mathcal{O}_{K_Z}}}.$$

Now, with the same replacements, the proof of [Wu25, Lem. 4.49] goes through verbatim. □

In summary,

Theorem 5.5 ([FC90], [Lan16], [MP19], and [Wu25])**.** *For abelian-type Shimura data, there are integral models satisfying Axiom 5.1 for all K at the beginning of §5.1.*

Remark 5.6. *In §5.2, we work over the generic fiber, so all arguments work without assumptions. In §5.3, we assume Axiom 5.1 holds for certain compactifications; however, the only condition we will need is a stratification with formal completions described by* ***toric schemes****. In §5.4, we will further impose Axiom 4.2 to define canonical integral models of compactifications; as we just mentioned, one does not have to require the stronger conditions in §5.4 to show the canonical extension of shtukas to integral models of toroidal compactifications.*

5.2. **Canonical extensions on generic fiber.**

Let (G,X) be a Shimura datum and Σ be smooth and projective with respect to K. On $\mathrm{Sh}^{\Sigma}_{K}(G,X)$, we will explain that the pro-Kummer-étale $\underline{\mathcal{G}^{c}(\mathbb{Z}_p)}$-torsor induces a log shtuka with a good description at the boundary.

5.2.1. Let $\mathbb{P}_K := \mathbb{P}_K(G,X)$ and $\mathbb{P}_{K_{\Phi}} := \mathbb{P}_{K_{\Phi}}(P_{\Phi},D_{\Phi})$ be the pro-étale torsor defined as in Definition 3.13.

Lemma 5.7. *Let* $\Phi_1 \xrightarrow{(\gamma,q_2)_K} \Phi_2$ *be an equivalence between cusp label representatives in* $\mathcal{CLR}(G,X)$ *(see* [MP19, §2.1.14]*). Then we have a canonical isomorphism* $\mathbb{P}_{K_{\Phi_1}} \to \mathbb{P}_{K_{\Phi_2}}$ *that makes the diagram commute:*

$$\begin{tikzcd}
\mathbb{P}_{K_{\Phi_1}} \arrow[r, "\cong"] \arrow[d] & \mathbb{P}_{K_{\Phi_2}} \arrow[d] \\
\mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1},D_{\Phi_1}) \arrow[r, "\cong"] & \mathrm{Sh}_{K_{\Phi_2}}(P_{\Phi_2},D_{\Phi_2}).
\end{tikzcd}$$

Similar statements hold for $\mathbb{P}_{\overline{K}_{\Phi_i}} \to \mathrm{Sh}_{\overline{K}_{\Phi_i}}(\overline{P}_{\Phi_i},\overline{D}_{\Phi_i})$ *and* $\mathbb{P}_{K_{\Phi_i,h}} \to \mathrm{Sh}_{K_{\Phi_i,h}}(G_{\Phi_i,h},D_{\Phi_i,h})$.

Proof. It suffices to show that, at each normal subgroup $K'_p \subset K_p$, we have a canonical dashed morphism fitting into the commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}_{K'_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \arrow[r, dashed, "\cong"] \arrow[d] & \mathrm{Sh}_{K'_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}) \arrow[d] \\
\mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \arrow[r, "\cong"] & \mathrm{Sh}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}),
\end{tikzcd}$$

where $K' = K'_p K^p$ and $K = K_p K^p$. Recall that $\gamma g_1 = q_2 g_2 k$ for some $k \in K$, we decompose $(\gamma, q_2)_K$ as

$$\Phi_1 \xrightarrow{(\gamma,1)_K} \Phi'_1 \xrightarrow{(1,q_2)_K} \Phi'_2 \xrightarrow{(1,k)_K} \Phi_2, \tag{5.3}$$

where $\Phi'_1 = (Q_{\Phi_2}, X^+_{\Phi_2}, \gamma g_1)$, $\Phi'_2 = (Q_{\Phi_2}, X^+_{\Phi_2}, g_2 k)$.

$(\gamma, 1)_K$: Conjugation by γ gives an isomorphism $[\mathrm{Int}(\gamma)] : \mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \to \mathrm{Sh}_{K_{\Phi'_1}}(P_{\Phi'_1}, D_{\Phi'_1})$, similarly at K'-level.

$(1, q_2)_K$: Since $(\gamma g_1) = (q_2)(g_2 k)$, $K_{\Phi'_2} = q_2^{-1} K_{\Phi'_1} q_2$, right multiplication by q_2 gives an isomorphism $[\cdot q_2] : \mathrm{Sh}_{K_{\Phi'_1}}(P_{\Phi'_1}, D_{\Phi'_1}) \to \mathrm{Sh}_{K_{\Phi'_2}}(P_{\Phi'_2}, D_{\Phi'_2})$, similarly at K'-level since $K'_{\Phi'_2} = q_2^{-1} K'_{\Phi'_1} q_2$.

$(1, k)_K$: Note that $K_{\Phi'_2} = K_{\Phi_2}$. The identity morphism induces $\mathrm{Sh}_{K_{\Phi'_2}}(P_{\Phi'_2}, D_{\Phi'_2}) \cong \mathrm{Sh}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2})$, similarly at K'-level since K' is normal in K and $K'_{\Phi'_2} = K'_{\Phi_2}$.

To summarize, we have a commutative diagram induced by (5.3):

(5.4)

$$\begin{tikzcd}
\mathrm{Sh}_{K'_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \arrow[r, "{[\mathrm{Int}(\gamma)]}"] \arrow[d] & \mathrm{Sh}_{K'_{\Phi'_1}}(P_{\Phi'_1}, D_{\Phi'_1}) \arrow[r, "{[\cdot q_2]}"] \arrow[d] & \mathrm{Sh}_{K'_{\Phi'_2}}(P_{\Phi'_2}, D_{\Phi'_2}) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Sh}_{K'_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}) \arrow[d] \\
\mathrm{Sh}_{K_{\Phi_1}}(P_{\Phi_1}, D_{\Phi_1}) \arrow[r, "{[\mathrm{Int}(\gamma)]}"] & \mathrm{Sh}_{K_{\Phi'_1}}(P_{\Phi'_1}, D_{\Phi'_1}) \arrow[r, "{[\cdot q_2]}"] & \mathrm{Sh}_{K_{\Phi'_2}}(P_{\Phi'_2}, D_{\Phi'_2}) \arrow[r, "\mathrm{id}"] & \mathrm{Sh}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2}).
\end{tikzcd}$$

The statements for $\mathbb{P}_{\overline{K}_{\Phi_i}} \to \mathrm{Sh}_{\overline{K}_{\Phi_i}}(\overline{P}_{\Phi_i}, \overline{D}_{\Phi_i})$ and $\mathbb{P}_{K_{\Phi_i,h}} \to \mathrm{Sh}_{K_{\Phi_i,h}}(G_{\Phi_i,h}, D_{\Phi_i,h})$ are proved in the same way. ☐

Take $\Phi_1 = \Phi_2 = \Phi$. Then γ belongs to

$$\Delta_{\Phi,K} := (\mathrm{Stab}_{Q(\mathbb{Q})}(D_\Phi) \cap P(\mathbb{A}_f) g_\Phi K g_\Phi^{-1}) / P(\mathbb{Q}).$$

For each $\gamma \in \Delta_{\Phi,K}$, we find a $q_2 \in P(\mathbb{A}_f)$ such that $\Phi \xrightarrow{(\gamma,q_2)_K} \Phi$. In particular, $\Delta_{\Phi,K}$ naturally acts on the tower

$$\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi) \to \mathrm{Sh}_{\overline{K}_\Phi}(\overline{P}_\Phi, \overline{D}_\Phi) \to \mathrm{Sh}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h}), \tag{5.5}$$

and by Lemma 5.7, $\Delta_{\Phi,K}$-action naturally lifts to the tower

$$\mathbb{P}_{K_\Phi} \to \mathbb{P}_{\overline{K}_\Phi} \to \mathbb{P}_{K_{\Phi,h}}. \tag{5.6}$$

Proposition 5.8. *The commutative diagrams (3.8) and (3.9) in Lemma 3.15 and Corollary 3.16 are equivariant under the $\Delta_{\Phi,K}$-action.*

Proof. We work with $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$; the other two follow in the same way. This essentially follows from Lemma 5.7, but we spell out the details for completeness. We abbreviate $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$ to Sh_{K_Φ}. Let $x \in \mathrm{Sh}_{K_\Phi}$ be a closed point. The supported shtuka $\mathrm{Sh}^\diamondsuit_{K_\Phi} \to \mathrm{Sht}_{\mathcal{P}^c_\Phi, \mu^c_\Phi, \delta=1, \mathrm{Spd}\, E}$ at x is determined by the Hodge–Tate period map $\mathrm{HT}_{\Phi,x} : \mathbb{P}_{K_\Phi,x} \to \mathrm{Gr}_{\mathcal{P}^c_\Phi, \mu^c_\Phi}$. This map is determined by the de Rham $\underline{\mathcal{P}^c_\Phi(\mathbb{Z}_p)}$-torsor $\mathbb{P}_{K_\Phi,x}$ itself, which in turn is determined by the Galois representation $\rho_x : \mathrm{Gal}(\overline{k(x)}|k(x)) \to \mathcal{P}^c_\Phi(\mathbb{Z}_p)$. We now recall the construction of ρ_x, following [KP23, §3.1].

Fix a lifting $x \in \mathrm{Sh}_{\varnothing_\Phi}(\overline{\mathbb{Q}}_p) := \mathrm{Sh}(P_\Phi, D_\Phi)(\overline{\mathbb{Q}}_p)$, with image $xK'_\Phi \in \mathrm{Sh}_{K'_\Phi}(\overline{\mathbb{Q}}_p)$ at each level $K'_\Phi \subset K_\Phi$. Let $S_{K'_\Phi,x}$ be the geometrically connected component of $\mathrm{Sh}_{K_\Phi'}$ containing xK'_Φ; it is defined over some abelian extension $E_{K'_\Phi,x}$ of E. For an open normal subgroup $K'_\Phi \subset K_\Phi$, we have Cartesian squares of Galois coverings under the right action of $K_\Phi/Z(P_\Phi)(\mathbb{Q})^-_{K_\Phi}K'_\Phi$:

$$\begin{tikzcd}
\mathrm{Sh}_{K'_\Phi} \arrow[d] & \mathrm{Sh}_{K'_\Phi} \otimes_E E_{K_\Phi,x} \arrow[l] \arrow[d] & \mathrm{Sh}_{K'_\Phi} \otimes_{\mathrm{Sh}_{K_\Phi}} S_{K_\Phi,x} \arrow[l, hook'] \arrow[d] & \widetilde{S}_{K'_\Phi,x} \arrow[l, hook'] \arrow[dl] \\
\mathrm{Sh}_{K_\Phi} & \mathrm{Sh}_{K_\Phi} \otimes_E E_{K_\Phi,x} \arrow[l] & S_{K_\Phi,x}, \arrow[l, hook'] &
\end{tikzcd} \tag{5.7}$$

where $\widetilde{S}_{K'_\Phi,x} \subset \mathrm{Sh}_{K'_\Phi} \otimes_E E_{K_\Phi,x}$ is the connected component that contains xK'_Φ. Let $\overline{x}$ be a geometric point over x, then the above diagram induces a morphism

$$\pi_1(S_{K_\Phi,x}, \bar{x}) \to \mathrm{Aut}(\widetilde{S}_{K'_\Phi,x}/S_{K_\Phi,x})^{\mathrm{op}} \subset K_\Phi/Z(P_\Phi)(\mathbb{Q})^-_{K_\Phi}K'_\Phi.$$

Taking the inverse limit, and specializing it to $(\operatorname{Spec} E_{K_\Phi,x}, \bar{s})$, we have

$$\widetilde{\rho}_x : \mathrm{Gal}(\overline{k(x)}|k(x)) \to \pi_1(S_{K_\Phi,x}, \bar{x}) \to K_\Phi/Z(P_\Phi)(\mathbb{Q})^-_{K_\Phi}.$$

The projection of $\widetilde{\rho}_x$ to p-factor $\mathcal{P}^c_\Phi(\mathbb{Z}_p)$ is ρ_x.

Let $\gamma \in \Delta_{\Phi,K}$. Let $y = \gamma x \in \mathrm{Sh}_{\varnothing_\Phi}(\overline{\mathbb{Q}}_p)$, as in Lemma 5.7, $[\gamma]$ acts compatibly on the diagram (5.7), thus $\widetilde{\rho}_x$ and $\widetilde{\rho}_y$ are conjugated by $[\gamma]$:

$$\begin{tikzcd}
\tilde{\rho}_x : \mathrm{Gal}(\overline{k(x)}|k(x)) \arrow[r] & \pi_1(S_{K_\Phi,x}, \bar{x}) \arrow[r] \arrow[d, "{[\gamma]}"'] & \mathrm{Aut}(\widetilde{S}_{K'_\Phi,x}/S_{K_\Phi,x})^{\mathrm{op}} \arrow[r] \arrow[d, "{[\gamma]}"'] & K_\Phi/Z(P_\Phi)(\mathbb{Q})^-_{K_\Phi}K'_\Phi \arrow[d, "\mathrm{Int}(q_2^{-1}\gamma)"'] \\
\tilde{\rho}_y : \mathrm{Gal}(\overline{k(y)}|k(y)) \arrow[r] & \pi_1(S_{K_\Phi,y}, \bar{y}) \arrow[r] & \mathrm{Aut}(\widetilde{S}_{K'_\Phi,y}/S_{K_\Phi,y})^{\mathrm{op}} \arrow[r] & K_\Phi/Z(P_\Phi)(\mathbb{Q})^-_{K_\Phi}K'_\Phi.
\end{tikzcd}$$

We fix a representation $\rho_\Phi : P_\Phi(\mathbb{Q}_p) \to P^c_\Phi(\mathbb{Q}_p) \to \mathrm{GL}(W_{\mathbb{Q}_p})$, and a lattice $W_{\mathbb{Z}_p} \subset W_{\mathbb{Q}_p}$ such that $\rho_\Phi(\mathcal{P}^c_\Phi(\mathbb{Z}_p)) \subset \mathrm{GL}(W_{\mathbb{Z}_p})$, this produces a de Rham local system $\mathbb{L}_{\rho_\Phi,W_{\mathbb{Z}_p}}$ as in (3.6). For simplicity, we denote $\mathbb{L} := \mathbb{L}_{\rho_\Phi,W_{\mathbb{Z}_p}}$. The canonical isomorphism $\gamma : \mathbb{P}_{K_\Phi,x} \to \mathbb{P}_{K_\Phi,y}$ induces an isomorphism $\gamma : \mathbb{L}_x \to \mathbb{L}_y$. We identify the underlying $\mathbb{Z}_p$-local systems $\mathbb{L}_x \to \mathbb{L}_y$, then the Galois action on $\mathbb{L}_y$ is twisted by $[\gamma]$-conjugation from the one on $\mathbb{L}_x$ since $\widetilde{\rho}_x$ and $\widetilde{\rho}_y$ are conjugated by $[\gamma]$. By the Cartan-Leray spectral sequence, the identification $\mathbb{L}_x \cong \mathbb{L}_y$ induces a canonical identification $(D_{\mathrm{dR}}(\mathbb{L}_x), \nabla_{\mathbb{L}_x}, \mathrm{Fil}_{\mathbb{L}_x}) \cong (D_{\mathrm{dR}}(\mathbb{L}_y), \nabla_{\mathbb{L}_y}, \mathrm{Fil}_{\mathbb{L}_y})$. In particular, in the construction [PR24, Prop. 2.6.3], $[\gamma]$-action identifies $\mathrm{DRT}(\mathbb{L}_x)$ and $\mathrm{DRT}(\mathbb{L}_y)$. Therefore, $[\gamma]$ induces an equivariant action on $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \to [\mathrm{Gr}_{P^c_\Phi,\mu^{c,-1}_\Phi} \underline{/\mathcal{P}^c_\Phi(\mathbb{Z}_p)}]$ where it acts trivially on the target. Finally, since $\Delta_{\Phi,K}$-action on Sh_{K_Φ} lifts canonically to a $\Delta_{\Phi,K}$-action on $\mathbb{P}_{K_\Phi}$ as in Lemma 5.7, we have a cocycle condition for the identifications among $\{\mathrm{DRT}(\mathbb{L}_{\gamma(x)})\}_{\gamma\in\Delta_{\Phi,K}}$, $\Delta_{\Phi,K}$ acts on the diagram (3.9). $\square$

5.2.2. Recall that we have

$$\mathfrak{X}^\circ_{\sigma,\eta} := \Delta^\circ_{\Phi,K}\backslash(\mathrm{Sh}_{K_\Phi}(\sigma))^\wedge_{\underset{\tau\in\Sigma^+(\Phi),\ \tau\subset\sigma}{\cup} \mathrm{Sh}_{K_\Phi,\tau}} \cong (\mathrm{Sh}^\Sigma_K)^\wedge_{\underset{\tau\in\Sigma^+(\Phi),\ \tau\subset\sigma}{\cup} \mathrm{Z}_{[(\Phi,\tau)]}}.$$

Here $\Delta^\circ_{\Phi,K} \subset \Delta_{\Phi,K}$ is the subgroup that stabilizes σ; when K is neat, it is independent of the choice of σ (see [Pin90, Thm. 6.19] and also[MP19, §2.1.19]).

We first consider the torsor defined by the tower $\varprojlim_{K'_p} \Delta^\circ_{\Phi,K'}\backslash\mathrm{Sh}_{K'_\Phi}$. Denote by ZP_Φ the connected component of $Z_G{\cdot}P_\Phi$, and by $ZP_\Phi(\mathbb{Q})_1 := \mathrm{Stab}_{ZP_\Phi(\mathbb{Q})}(D_\Phi)$. When K is neat, by [Wu25, Lem. 1.11], we have that

$$\Delta^\circ_{\Phi,K}\backslash\mathrm{Sh}_{K_\Phi}(\mathbb{C}) = ZP_\Phi(\mathbb{Q})_1\backslash D_\Phi \times ZP_\Phi(\mathbb{Q})_1P_\Phi(\mathbb{A}_f)/ZP_\Phi(\mathbb{Q})_1P_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}.$$

Then the Galois group is $\mathscr{G}_p^\circ(\Phi, K) := \mathrm{Gal}(\varprojlim_{K'_p \subset K_p} \Delta^\circ_{\Phi,K'} \backslash \mathrm{Sh}_{K'_\Phi} / \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi})$

$$\begin{aligned}
&= \mathrm{Gal}(\frac{\varprojlim_{K'_p \subset K_p} ZP_\Phi(\mathbb{Q})_1 \backslash D_\Phi \times ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) / ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) \cap g_\Phi K' g_\Phi^{-1}}{ZP_\Phi(\mathbb{Q})_1 \backslash D_\Phi \times ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) / ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}}) \\
&= \frac{ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}}{(Z(\mathbb{Q})^- \cap g_\Phi K g_\Phi^{-1})(ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) \cap g_\Phi K^p g_\Phi^{-1})} \\
&= \frac{ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) g_\Phi K^p g_\Phi^{-1} \cap g_\Phi K_p g_\Phi^{-1}}{(Z(\mathbb{Q})^- \cap g_\Phi K g_\Phi^{-1}) g_\Phi K^p g_\Phi^{-1} \cap g_\Phi K_p g_\Phi^{-1}} \\
&= g_\Phi \frac{(g_\Phi^{-1} ZP_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) g_\Phi) K^p \cap K_p}{(Z(\mathbb{Q})^- \cap K) K^p \cap K_p} g_\Phi^{-1}.
\end{aligned} \tag{5.8}$$

The image of the numerator of $\mathscr{G}_p^\circ(\Phi, K)$ in $ZP^c_\Phi(\mathbb{Q}_p) = (ZP_\Phi)^c(\mathbb{Q}_p)$ lies in $P^*_\Phi(\mathbb{Q}_p)$; this is also due to neatness and the fact that ZP^c_Φ / P^*_Φ is a cuspidal torus.

Definition 5.9. *Denote* $\mathbb{P}^*_{K_\Phi} := \varprojlim_{K'_{\Phi,p}} \Delta^\circ_{\Phi,K'} \backslash \mathrm{Sh}_{K'_\Phi} \times^{\mathscr{G}_p^\circ(\Phi,K)} \underline{\mathcal{P}^*_\Phi(\mathbb{Z}_p)}$.

Then, by Proposition 5.8, the quotient induced by the action of $\Delta^\circ_{\Phi,K}$ descends $\mathbb{P}_{K_\Phi} \times^{\mathcal{P}^c_\Phi(\mathbb{Z}_p)} \underline{\mathcal{P}^*_\Phi(\mathbb{Z}_p)}$ to $\mathbb{P}^*_{K_\Phi}$; the Hodge-Tate map HT_{K_Φ} descends to

$$\mathrm{HT}^*_{K_\Phi} : \mathbb{P}^*_{K_\Phi} \to \mathrm{Gr}_{P^*_\Phi, \mu^{*,-1}_\Phi},$$

where μ^*_Φ is the projection of μ_Φ to P^*_Φ. More precisely,

Lemma 5.10. *The commutative diagram (3.9) descends to the commutative diagram:*

$$\begin{tikzcd}
\Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\diamondsuit \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{\overline{K}_\Phi}(\overline{P}_\Phi, \overline{D}_\Phi)^\diamondsuit \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^\diamondsuit \arrow[d] \\
\mathrm{Sht}_{\mathcal{P}^*_\Phi, \mu^*_\Phi, \delta=1, \mathrm{Spd}\, E} \arrow[r] & \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi, \bar{\mu}^*_\Phi, \delta=1, \mathrm{Spd}\, E} \arrow[r] & \mathrm{Sht}_{\mathcal{G}^*_{\Phi,h}, \mu^*_{\Phi,h}, \delta=1, \mathrm{Spd}\, E}.
\end{tikzcd} \tag{5.9}$$

Proof. This follows from Proposition 5.8, Lemma 1.48 and Proposition 1.45. ☐

Let $\mathfrak{W} = \mathrm{Spf}(R, I) \subset \mathfrak{X}^\circ_{\sigma,\eta}$ be an affine open formal subscheme, we can consider the flat morphisms $W = \mathrm{Spec}\, R \to \mathrm{Sh}^\Sigma_K$ and $W \to \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi}(\sigma)$. Let $W^0 \subset W$ be the common open subscheme associated with $\mathrm{Sh}_K \subset \mathrm{Sh}^\Sigma_K$ and $\Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi} \subset \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi}(\sigma)$. We have flat morphisms $W^0 \to \mathrm{Sh}_K(G, X)$ and $W^0 \to \Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$.

We pull $\mathbb{P}_K$ and $\mathbb{P}^*_{K_\Phi}$ back to W^0, and denote them by $\mathbb{P}_{K,W^0}$ and $\mathbb{P}_{K_\Phi,W^0}$, respectively. Now we compare these two pro-étale torsors.

Lemma 5.11. $\mathbb{P}_{K,W^0} \cong \underline{\mathcal{G}^c(\mathbb{Z}_p)} \times^{\mathcal{P}^*_\Phi(\mathbb{Z}_p)} \mathbb{P}^*_{K_\Phi,W^0}$, *where* $\mathcal{P}^*_\Phi(\mathbb{Z}_p) \to \mathcal{G}^c(\mathbb{Z}_p)$ *is the composition of* $\mathcal{P}^*_\Phi(\mathbb{Z}_p) \xrightarrow{\mathrm{Int}(g_\Phi^{-1})} g_\Phi^{-1} \mathcal{P}^*_\Phi(\mathbb{Z}_p) g_\Phi \hookrightarrow \mathcal{G}^c(\mathbb{Z}_p)$.

Proof. Note that $g_\Phi^{-1} \mathcal{P}^*_\Phi(\mathbb{Z}_p) g_\Phi \to \mathcal{G}^c(\mathbb{Z}_p)$ is injective by Definition 3.11. Let $K' \subset K$. The cone decomposition of the toroidal compactification at level K' is the cone decomposition induced by Σ, which is denoted by the same symbol. There is a transition map $\pi^\Sigma_{K',K} : \mathrm{Sh}^\Sigma_{K'} \to \mathrm{Sh}^\Sigma_K$. Pick a stratum $\mathrm{Z}_{[(\Phi,\sigma)],K}$. The preimage of $\mathrm{Z}_{[(\Phi,\sigma)],K}$ under $\pi^\Sigma_{K',K}$ is the disjoint union

$$\coprod_{[(\Phi',\sigma')] \to [(\Phi,\sigma)]} \mathrm{Z}_{[(\Phi',\sigma')],K'},$$

where $[(\Phi', \sigma')]$ are cusp labels with cones in $\mathrm{Cusp}_{K'}(G, X, \Sigma)$ that map to $[(\Phi, \sigma)]$ in $\mathrm{Cusp}_K(G, X, \Sigma)$.

Let $\mathfrak{W} = \operatorname{Spf} R \subset \mathfrak{X}^{\circ}_{\sigma,\eta}$ be an open formal subscheme. Let $\mathfrak{W}' = \operatorname{Spf} R'$ be the pullback via $\pi^{\Sigma}_{K',K}$. Denote $W := \operatorname{Spec} R$ and $W' := \operatorname{Spec} R'$. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}^{\Sigma}_{K'} \arrow[d] & W' \arrow[l] \arrow[r] \arrow[d] & \coprod_{[(\Phi',\sigma')]\to[(\Phi,\sigma)]} \Delta^{\circ}_{\Phi',K'}\backslash \mathrm{Sh}_{K_{\Phi'}}(\sigma') \arrow[d] \\
\mathrm{Sh}^{\Sigma}_{K} & W \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathrm{Sh}_{K_{\Phi}}(\sigma),
\end{tikzcd}$$

where both squares are Cartesian.

Recall that, fixing Q_Φ, the set $I_K(Q_\Phi, \Sigma)$ (see [Wu25, Def. 1.32]) consists of the cusp labels with cones $[(\Phi_1, \sigma_1)]$ equivalent to $[(Q_\Phi, X^{+,\prime}_{\Phi}, g'_\Phi, \sigma')]$ in $\mathrm{Cusp}_K(G, X, \Sigma)$ for some $(X^{+,\prime}_{\Phi}, g'_\Phi, \sigma')$; we add subscript K here to emphasize the role of K in the equivalence relation. By [Wu25, Prop. 1.4] and the paragraph above [Wu25, Def. 1.16],

$$I_K(Q_\Phi, \Sigma) \cong \coprod_{[g]\in \mathrm{Stab}_{Q_\Phi(\mathbb{Q})}(D_\Phi) P_\Phi(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K} \mathrm{Stab}_{Q_\Phi(\mathbb{Q})}(D_\Phi)\backslash[\Sigma^+(Q_\Phi, g) \times P_\Phi(\mathbb{A}_f)\backslash P_\Phi(\mathbb{A}_f) gK/K].$$

From this, we see that K permutes all $[(\Phi', \sigma')]$ mapping to $[(\Phi, \sigma)]$. Restricting to W^0, the diagram above gives

$$\begin{tikzcd}
\mathrm{Sh}_{K'} \arrow[d] & W'^{,0} \arrow[l] \arrow[r] \arrow[d] & \coprod_{[(\Phi',\sigma')]\to[(\Phi,\sigma)]} \Delta^{\circ}_{\Phi',K'}\backslash \mathrm{Sh}_{K_{\Phi'}} \arrow[d] \\
\mathrm{Sh}_{K} & W^0 \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathrm{Sh}_{K_{\Phi}},
\end{tikzcd}$$

where both squares are Cartesian. Now, we shrink K_p and take the inverse limit. The pushout of the inverse limit of $([(\Phi, \sigma)], K') \to ([(\Phi, \sigma)], K)$ on the right vertical arrow gives rise to $\mathbb{P}^*_{K_\Phi}$.

We need to see that on W^0, $\mathbb{P}_{K,W^0} \cong \underline{\mathcal{G}^c(\mathbb{Z}_p)} \times^{\underline{\mathcal{P}^*_\Phi(\mathbb{Z}_p)}} \mathbb{P}^*_{K_\Phi,W^0}$. For this, we compute that K_p acts transitively on the fiber of $\varprojlim_{K'_p} I_{K'}(Q_\Phi, \Sigma)$ at $[(\Phi, \sigma)]$ with stabilizer $(g_\Phi^{-1} \mathrm{Stab}_{Q_\Phi(\mathbb{Q})}(D_\Phi, \sigma) P_\Phi(\mathbb{A}_f) g_\Phi K^p) \cap K_p$. When K is neat, this intersection is independent of the choice of $\sigma \in \Sigma^+(\Phi)$ and is equal to $(g_\Phi^{-1} Z P_\Phi(\mathbb{Q})_1 P_\Phi(\mathbb{A}_f) g_\Phi K^p) \cap K_p$.

We finally compute that on W^0: $[\varprojlim \Delta^{\circ}_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}] \times^{\mathscr{G}^{\circ}_p(\Phi,K)} \underline{\mathcal{G}^c(\mathbb{Z}_p)}$

$$\begin{aligned}
&= [\varprojlim \Delta^{\circ}_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}] \times^{\mathscr{G}^{\circ}_p(\Phi,K)} \underline{K_p/(K_p \cap K^p Z(\mathbb{Q})^-_K)} \times^{\underline{K_p/(K_p\cap K^p Z(\mathbb{Q})^-_K)}} \underline{\mathcal{G}^c(\mathbb{Z}_p)} \\
&= [\varprojlim_{K'_p\subset K_p} \coprod_{[(\Phi',\sigma')]\to[(\Phi,\sigma)]} \Delta^{\circ}_{\Phi',K'}\backslash \mathrm{Sh}_{K_{\Phi'}}] \times^{\underline{K_p/(K_p\cap K^p Z(\mathbb{Q})^-_K)}} \underline{\mathcal{G}^c(\mathbb{Z}_p)} \\
&= \varprojlim_{K'_p\subset K_p} \mathrm{Sh}_{K'} \times^{\underline{K_p/(K_p\cap K^p Z(\mathbb{Q})^-_K)}} \underline{\mathcal{G}^c(\mathbb{Z}_p)}.
\end{aligned}$$

The only nontrivial part is the computation from the second line to the third line. This is done by comparing (5.8) with the stabilizer in the last paragraph; note that there is a g_Φ-conjugation difference in the construction. $\square$

Lemma 5.12. *We have the following commutative diagram*

$$\begin{tikzcd}
\mathbb{P}^*_{K_\Phi,W^0} \arrow[r] \arrow[d, "\mathrm{HT}^*_{K_\Phi}"'] & \mathbb{P}_{K,W^0} \arrow[d, "\mathrm{HT}_K"'] \\
\mathrm{Gr}_{P^*_\Phi,\mu^{*,-1}_\Phi} \arrow[r, "\mathrm{Int}(g_\Phi^{-1})"] & \mathrm{Gr}_{G^c,\mu^{c,-1}}\,.
\end{tikzcd}$$

Proof. Recall that, in (3.6), for any representation $\rho : G^c(\mathbb{Q}_p) \to \mathrm{GL}(V_{\mathbb{Q}_p})$ and a lattice $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ such that $\rho(\mathcal{G}^c(\mathbb{Z}_p)) \subset \mathrm{GL}(V_{\mathbb{Z}_p})$, ρ induces a morphism $\rho_\Phi : P^*_\Phi(\mathbb{Q}_p) \to \mathrm{GL}(V_{\mathbb{Q}_p})$ such that $\rho_\Phi(\mathcal{P}^*_\Phi(\mathbb{Z}_p)) = \rho(g_\Phi^{-1}(\mathcal{P}^*_\Phi(\mathbb{Z}_p))g_\Phi) \subset \mathrm{GL}(V_{\mathbb{Z}_p})$. Since $\mathbb{L}_{\rho_\Phi, V_{\mathbb{Z}_p}, W^0} = \mathbb{L}_{\rho(g_\Phi^{-1}(-)g_\Phi), V_{\mathbb{Z}_p}, W^0}$ is de Rham by [LZ17], the association of Hodge-Tate period map is intrinsic (see [PR24, Prop. 2.6.3]). This gives the desired commutative diagram. ☐

Corollary 5.13. *We have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Sh}_K(G,X)^\lozenge \arrow[d] & W^{0,\lozenge} \arrow[l] \arrow[r] & \Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}^c,\mu^c} & & \mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi}. \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"']
\end{tikzcd} \tag{5.10}$$

Corollary 5.14. *For any representation $(\rho, V_{\mathbb{Z}_p})$ of $\mathcal{G}^c(\mathbb{Z}_p)$ where $V_{\mathbb{Z}_p}$ is a finite free module over $\mathbb{Z}_p$, the pro-Kummer étale local system $J^\Sigma_* \mathbb{L}_{\rho, V_{\mathbb{Z}_p}, W^0}$ has unipotent geometric monodromy in the sense of Remark 2.44 (cf.* [DLLZ23a, Def. 6.3.7]*). Hence, by* [LZ17] *and Corollary 2.53 (2) and (3), there is a log shtuka $\mathscr{P}^{\mathrm{can}}_\eta$ on $(\mathrm{Sh}^\Sigma_K)^{\log \lozenge}$ uniquely extending the shtuka $\mathscr{P}_\eta$ on $(\mathrm{Sh}_K(G,X))^\lozenge$ associated with $\mathbb{P}_K$ under* [PR24, Prop. 2.5.3].

Proof. Note that it suffices to consider the mixed Shimura varieties in the form of

$$\varprojlim_{K'_p \subset K_p} \mathrm{Sh}_{\widetilde{K}'_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi),$$

where $\widetilde{K}'_\Phi := ZP_\Phi(\mathbb{A}_f) \cap g_\Phi K' g_\Phi^{-1}$. Indeed, this follows from the description of the boundary using ZP-cusps (see [Wu25, (1.32)]), and from the open and closed immersion $\Delta^\circ_{\Phi,K'}\backslash \mathrm{Sh}_{K'_\Phi} \hookrightarrow \mathrm{Sh}_{\widetilde{K}'_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi)$ for all K'_p by [Wu25, Lem. 1.42 (1)] and [Wu25, Lem. 1.11]. For unipotence, combine Lemma 5.11 and Lemma 3.27 (the mixed Shimura variety in the lemmas is $\mathrm{Sh}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi)$). Then the action of the geometric Kummer étale fundamental group factors through $\mathbf{E}'_\infty$ on the pro-Kummer étale torsor $J(\sigma)_* \mathbb{P}^*_{K_\Phi}$ on $\Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)(\sigma) \subset \mathrm{Sh}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi)(\sigma)$. Here $J(\sigma) : \Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi) \hookrightarrow \Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)(\sigma)$ is the twisted toric embedding. By the proof of Lemma 3.27 and since we are considering P^*_Φ, which is a quotient from P^c_Φ, we can further reduce the problem to the case where $P = P^c$. The torus action is defined by (3.14) and (3.15). When K_p varies, the action of the torus part is given by the action of a conjugate of a unipotent element $u_f \in U(\mathbb{A}_f)$ as explained at the beginning of §3.4.3. The other statement is self-explanatory. ☐

5.3. **Canonical extensions on integral models.** We formulate and prove two general extension results for log shtukas. The techniques we use are consequences of toric charts but not the geometry of integral models of (open) Shimura varieties; this, in particular, enables us to show the main theorems in satisfactory generality.

5.3.1. *Gluing lemma.* The rough idea of the following lemma is to glue the log shtukas using toric charts. In particular, we do not use the geometry of $\mathscr{S}_K$. Let E be the completion of the reflex field of the Shimura datum (G, X) that we consider.

Lemma 5.15. *We still assume that Σ is smooth projective. Suppose that there is an integral model $\mathscr{S}^\Sigma_K$ satisfying Axiom 5.1 for (G, X, K). (In fact, we only need the assumptions for toroidal compactifications.)*

Choose any affine formal open subscheme $\mathfrak{W} = \mathrm{Spf}(A, I) \subset \mathfrak{X}_\Upsilon$. Let $W := \mathrm{Spec}\, A$. The open embedding $j : \mathscr{S}_K \hookrightarrow \mathscr{S}^\Sigma_K$ induces an open embedding $j_W : W^0 := \mathscr{S}_K \times_{j, \mathscr{S}^\Sigma_K} W \hookrightarrow W$, which fits *into the following commutative diagram*

$$\begin{tikzcd} W^0 \arrow[r, hook, "j_W"] \arrow[d, "i^0"] & W \arrow[d, "i"] \\ \mathscr{S}_K \arrow[r, hook, "j"] & \mathscr{S}_K^{\Sigma}. \end{tikzcd}$$

Denote $\mathscr{S} := \mathscr{S}_K$ *and* $\mathcal{X} := \mathscr{S}_K^{\Sigma}$ *for simplicity. We further assume:*

- *There is a* $\mathcal{G}^c$*-shtuka* $\mathscr{P}_{\mathscr{S}}$ *on* $\mathscr{S}^{\Diamond/}$ *extending the one* $\mathscr{P}_\eta := \mathscr{P}_{\mathscr{S}}|_{\mathscr{S}_E}$ *on the generic fiber. Assume that* $\mathscr{P}_\eta$ *extends (uniquely) to a log shtuka* $\mathscr{P}_\eta^{\mathrm{can}}$ *on* $\mathcal{X}_E^{\log\Diamond}$*. Then there is a canonical isomorphism* $\theta_\eta : i_E^{0,*}\mathscr{P}_\eta \cong i_E^{0,*} j_E^* \mathscr{P}_\eta^{\mathrm{can}} \xrightarrow{\sim} j_{W,E}^* i_E^* \mathscr{P}_\eta^{\mathrm{can}}$ *on* $W_E^{0,\Diamond}$*.*
- *For any* $\mathfrak{W}$*, there is a log* $\mathcal{G}^c$*-shtuka* $\mathscr{P}_W$ *on* $W^{\log\diamond}$ *extending* $i_E^*\mathscr{P}_\eta^{\mathrm{can}}$*, such that there is a unique isomorphism*
  $$\theta : i^{0,*}\mathscr{P}_{\mathscr{S}} \xrightarrow{\sim} j_W^* \mathscr{P}_W$$
  on $W^{0,\diamond}$ *extending* θ_η*.*

Then there is a unique log $\mathcal{G}^c$*-shtuka* $\mathscr{P}^{\mathrm{can}}$ *on* $\mathcal{X}^{\log\diamond} = \mathcal{X}^{\log\Diamond}$ *extending both* $\mathscr{P}_\eta^{\mathrm{can}}$ *and* $\mathscr{P}_{\mathscr{S}}$*.*

Proof. Induction on the dimension of σ in the pair $\Upsilon = [(\Phi, \sigma)]$. Note that $\dim \sigma = \operatorname{codim} \mathcal{Z}_{\Upsilon,K}$. When σ is trivial, by assumption, there is a shtuka $\mathscr{P}_{\mathscr{S}}$.

Assume that there is an extension of log shtuka $\mathscr{P}^k$ on $\mathcal{X}^{k,\log\Diamond/}$, whose underlying scheme $\mathcal{X}^k$ is set-theoretically the union of all strata $\mathcal{Z}_\Upsilon$ of codimension $\leq k$. The strata of codimension exactly $k+1$ are disjoint in $\mathcal{X}$. So we add one more stratum $\Upsilon' := [(\Phi', \sigma')]$ to $\mathcal{X}^k$ and show that there is an extension of the log shtuka on $(\mathcal{X}^k \cup \mathcal{Z}_{\Upsilon'})^{\log\Diamond/}$ or $(\mathcal{X}^k \cup \mathcal{Z}_{\Upsilon'})^{\log\diamond}$. (Here $\mathcal{X}^k \cup \mathcal{Z}_{\Upsilon'}$ denotes the normal subscheme in $\mathcal{X}$.)

Pick any affine open subscheme V of $\mathcal{X}$ that is contained in $\mathcal{X}^k \cup \mathcal{Z}_{\Upsilon'}$ and intersects with $\mathcal{Z}_{\Upsilon'}$. After taking a suitable étale cover and taking an affine open subscheme again, there is an affine scheme $U = \operatorname{Spec} B$ and étale morphisms $\mathsf{e}_1 : U \to V = \operatorname{Spec} A$ and $\mathsf{e}_2 : U \to T = \operatorname{Spec} R \times \mathbf{E}_{K_{\Phi'}}(\sigma')$ for some normal flat R, such that the pullbacks of the stratifications on V and T via e_1 and e_2 coincide on U. Indeed, this follows from a refinement of Artin's approximation theorem (see [Wu25, Prop. 4.53]). We can choose finitely many such U covering $\mathcal{Z}_{\Upsilon'}$.

Denote by τ_1, ..., τ_{k+1} the 1-dimensional faces of σ'. Denote by D_i the divisor on U defined by $\mathsf{e}_2^{-1}(T_{\tau_i})$ for $1 \leq i \leq (k+1)$; denote by U_i the complement $U \backslash D_i$. Denote $U' := \bigcup_{i=1}^{k+1} U_i$.

By construction, D_i is defined by a principal ideal $I_i = (s_i) \subset B$. Denote by I the ideal generated by I_i for all i. Denote by $\mathbf{V}$ the completion of V along the closed stratum $Z \subset V$ corresponding to $T_{\sigma'}$; the stratum Z corresponds to an ideal J. Then $\operatorname{Spec} \widehat{A}_J \times_V U = \operatorname{Spec} \widehat{B}_I$ and $\mathsf{e}_1^{-1} V(J) = V(I) = \mathsf{e}_2^{-1} T_{\sigma'}$. Denote $\mathbf{U} := \operatorname{Spec} \widehat{B}_I$ and $\mathbf{U}' := \operatorname{Spec} \widehat{B}_I \backslash V(I)$.

We now claim that there is an isomorphism between v-sheaves

$$(U')^{\log\diamond} \coprod_{(\mathbf{U}')^{\log\diamond}} \mathbf{U}^{\log\diamond} \cong U^{\log\diamond}. \tag{5.11}$$

Here, the LHS is defined as the v-sheaf gluing $(U')^{\log\diamond}$ and $\mathbf{U}^{\log\diamond}$. The map from left to right is injective. Since both side are v-sheaves by Theorem 2.18, for any affinoid $S^\sharp \in \mathrm{Perfd}$, it suffices to check surjectivity over a v-cover. For this, let $Z = \operatorname{Spa}(\prod_{t\in\mathcal{T}} C_t^+[1/\varpi_t], \prod_{t\in\mathcal{T}} C_t^+) \to S^\sharp$ be a v-cover constructed by a product of points, where $C_t^+[1/\varpi_t]$ are complete algebraically closed non-archimedean fields and C_t^+ are open and bounded valuation rings (see [Gle25, Ex. 1.1, Def. 1.2]). This cover is a strictly totally disconnected perfectoid space (see [Gle25, Prop. 1.6]), so all points are analytic.

For each C_t^+, if there is a map $f : \operatorname{Spec} C_t^+ \to U$, then the map factors through either $\mathbf{U}$ or U' according to the valuations of $\{s_i\}$ are all in the maximal ideal of C_t^+ or not: Indeed, for any

$(C_t^+[1/\varpi_t], C_t^+)$, the valuation $v_t : C_t^+[1/\varpi_t] \to \mathbb{R}_{\geq 0}$ is given by the ϖ_t-topology by [SW20, Lem. 4.2.2, Def. 4.2.3]. Therefore, for any $s_i \in C_t^+$, s_i lies in the maximal ideal if and only if $v_t(s_i) < 1$ if and only if the whole $\widehat{B}_{(s_i)}$ maps into C_t^+. If $v_t(s_i) = 1$ then $s_i^{-1} \in C_t^+$.

For any $(S^\sharp, \mathcal{M}_{S^\sharp}) \in U^{\log\diamond}$, we pullback the log structure to Z and denote it by $\mathcal{M}$. Then the log structure will also factor through U' or $\mathbf{U}$, since the log structures on U' and $\mathbf{U}$ are pulled back from the one on U. The claim is proved.

Note that there is a log shtuka $\mathscr{P}_{\mathbf{U}}$ on $(\mathbf{U})^{\log\diamond}$ given by pulling back the log shtuka $\mathscr{P}_{\mathbf{V}}$ on $(\mathbf{V})^{\log\diamond}$ and a log shtuka $\mathscr{P}^k_{U'}$ given by pulling back $\mathscr{P}^k$. The restrictions of both $\mathscr{P}^k_{U'}$ and $\mathscr{P}_{\mathbf{U}}$ to $(\mathbf{U}')^{\log\diamond}$ both extend the pullback of $\mathscr{P}^{\mathrm{can}}_\eta$ to $(\mathbf{U}'_{\mathbb{Q}_p})^{\log\Diamond}$. Now it follows from Theorem 2.55 that there is a canonical isomorphism between $\mathscr{P}^k_{U'}$ and $\mathscr{P}_{\mathbf{U}}$ on $(\mathbf{U}')^{\log\diamond}$. (Here we can use Theorem 2.55 because $\mathbf{U}'$ is a finite union of spectra of excellent Noetherian normal flat domains; for the fact that the completion is still excellent in this case, see [KS21, Appendix A], especially the table in the end.) Combining (5.11) in the last paragraph with Proposition 2.34, we extend the log shtuka to U. Taking an étale cover of U's constructed as the third paragraph and by Theorem 2.55 and the étale descent of shtukas, we obtain an extension $\mathscr{P}^{k,\Upsilon'}$ on $(\mathcal{X}^k \cup \mathcal{Z}_{\Upsilon'})^{\log\diamond}$.

Now, we choose another stratum $\Upsilon'' = [(\Phi'', \sigma'')]$ with codimension $(k+1)$ and repeat the construction above. Since there are only finitely many such strata and finitely many choice of k, we finally construct an extension on $(\mathcal{X}^{k+1})^{\log\diamond}$, and we are done by induction. □

5.3.2. *Extension along an affine toric embedding.* Fix $\mathbf{E}$ a split torus over $\mathbb{Z}$. Let $N = \mathbf{X}_*(\mathbf{E})$ and $M = \mathbf{X}^*(\mathbf{E})$. Let $\sigma \subset N \otimes \mathbb{R}$ be a smooth convex polyhedral cone, and $\sigma^\vee \subset M \otimes \mathbb{R}$ be its dual. We consider the toric embedding $\mathbf{E} \hookrightarrow \mathbf{E}(\sigma) = \operatorname{Spec} \mathbb{Z}[\sigma^\vee \cap M]$. We can naturally endow $\mathbf{E}(\sigma)$ with a log structure associated with $D_\sigma = \mathbf{E}(\sigma) \backslash \mathbf{E} \hookrightarrow \mathbf{E}(\sigma)$, and we denote it by $(\mathbf{E}(\sigma), \mathcal{M}_\sigma)$, with chart $\mathsf{P}_\sigma \to \mathcal{M}_\sigma$ associated with the monoid $\sigma^\vee \cap M/(\sigma^\vee \cap M)^\times$.

Let Y be a normal scheme that is separated, flat, and of finite type over $\mathbb{Z}_p$; we also assume that the generic fiber of Y is smooth.

Let X be an $\mathbf{E}$-torsor over Y. The twisted toric embedding $X \hookrightarrow X(\sigma) := X \times^{\mathbf{E}} \mathbf{E}(\sigma)$ is equipped with a log structure associated with the chart P_σ.

Let $\mathbf{E}'_\infty := \varprojlim_{x \mapsto x^p} \mathbf{E} = \varprojlim \mathbf{E}_n$, where $\mathbf{E}_n$ denotes $\operatorname{Spec} \mathbb{Z}[\frac{1}{p^n} M]$. Write $\mathbf{E}_n(\sigma) = \operatorname{Spec} \mathbb{Z}[\sigma^\vee \cap \frac{1}{p^n} M]$ and $\mathbf{E}'_\infty(\sigma) = \operatorname{Spec} \mathbb{Z}[\sigma^\vee \cap M[\frac{1}{p}]]$.

Note that $X \hookrightarrow X(\sigma)$ is equipped with an equivariant $\mathbf{E}$-action over Y. We write this action as $\mathbb{E} : \mathbf{E} \times X \to X$ and $\mathbb{E}(\sigma) : \mathbf{E} \times X(\sigma) \to X(\sigma)$. Pre-composing with $p_\infty \times \mathrm{id} : \mathbf{E}'_\infty \times X \to \mathbf{E} \times X$ (resp. $p_\infty \times \mathrm{id} : \mathbf{E}'_\infty \times X(\sigma) \to \mathbf{E} \times X(\sigma)$), where $p_\infty : \mathbf{E}'_\infty \to \mathbf{E}$ is the canonical projection, we obtain an $\mathbf{E}'_\infty$-action $\mathbb{E}_\infty : \mathbf{E}'_\infty \times X \to X$ (resp. $\mathbb{E}_\infty(\sigma) : \mathbf{E}'_\infty \times X(\sigma) \to X(\sigma)$); the actions $\mathbb{E}_\infty$ and $\mathbb{E}_\infty(\sigma)$ are equivariant with respect to $X \hookrightarrow X(\sigma)$.

Construction 5.16. The cover $X \to Y$ is faithfully flat. There is a canonical isomorphism between schemes $i : \mathbf{E} \times X \xrightarrow{\sim} X \times_Y X$; for any $\mathbb{Z}_p$-algebra R, $e \in \mathbf{E}(R)$, $x \in X(R)$, and $y := e \cdot x$, i sends (e, x) to (x, y). The descent datum $\varphi : X \times_Y X \xrightarrow{\sim} X \times_Y X$ for the cover $X \to Y$ is an isomorphism $\varphi = ((-\mathrm{id}) \circ p_1 \times \mathbb{E}) : \mathbf{E} \times X \xrightarrow{\sim} \mathbf{E} \times X$ after pulled back via i.

Let $i_\infty := i \circ (p_\infty \times \mathrm{id})$. We also pull back φ to an isomorphism of $\mathbf{E}'_\infty \times X$; i.e., we define $\varphi_\infty := ((-\mathrm{id}) \circ p_1 \times \mathbb{E}_\infty) : \mathbf{E}'_\infty \times X \xrightarrow{\sim} \mathbf{E}'_\infty \times X$.

For toric embeddings, we can also define

$$\varphi_\infty(\sigma) := ((-\mathrm{id}) \circ p_1 \times \mathbb{E}_\infty(\sigma)) : \mathbf{E}'_\infty \times X(\sigma) \xrightarrow{\sim} \mathbf{E}'_\infty \times X(\sigma).$$

Similarly, we define $\varphi(\sigma)$ by projection to $\mathbf{E}$. The action of $\mathbf{E}$ on $X(\sigma)$ is not free if $\sigma^\vee \cap M/(\sigma^\vee \cap M)^\times$ is nontrivial; in this case $\varphi(\sigma)$ is not an fpqc descent datum. □

Definition 5.17. *Let $(\mathcal{G}, \mu)$ be as in §1.3, where $\mathcal{G}$ is a quasi-parahoric model of a possibly non-reductive linear algebraic group (it was denoted as $\mathcal{P}$ in §1.3). Let $(\mathscr{P}, \phi_{\mathscr{P}})$ be a $\mathcal{G}$-shtuka (bounded* *by μ) on $X^\diamond$. We say that there is an (equivariant) $\mathbf{E}'_\infty$-action on $\mathscr{P}$ if there is an isomorphism*

$$\varphi(\mathscr{P}) : p_2^*(\mathscr{P}, \phi_{\mathscr{P}}) \xrightarrow{\sim} \mathbb{E}_\infty^*(\mathscr{P}, \phi_{\mathscr{P}})$$

on $(\mathbf{E}'_\infty \times X)^\diamond$ that satisfies the following conditions:

(1) Let $s : X \to \mathbf{E}'_\infty \times X$ be the identity section of $\mathbf{E}'_\infty$. Then $s^*\varphi(\mathscr{P}) = \mathrm{id}_{(\mathscr{P},\phi_{\mathscr{P}})}$;

(2) There is a commutative diagram (we omit $\phi_{\mathscr{P}}$ below):

$$\begin{tikzcd}
p_{23}^* p_2^* \mathscr{P} \arrow[r, "p_{23}^*\varphi(\mathscr{P})"] \arrow[d, equal] & p_{23}^* \mathbb{E}_\infty^* \mathscr{P} = (\mathrm{id} \times \mathbb{E}_\infty)^* p_2^* \mathscr{P} \arrow[r, "(\mathrm{id}\times\mathbb{E}_\infty)^*\varphi(\mathscr{P})"] & (\mathrm{id} \times \mathbb{E}_\infty)^* \mathbb{E}_\infty^* \mathscr{P} \arrow[d, equal] \\
(m \times \mathrm{id})^* p_2^* \mathscr{P} \arrow[rr, "(m\times\mathrm{id})^*\varphi(\mathscr{P})"] & & (m \times \mathrm{id})^* \mathbb{E}_\infty^* \mathscr{P}
\end{tikzcd} \tag{5.12}$$

over the commutative diagram

$$\begin{tikzcd}
\mathbf{E}'_\infty \times \mathbf{E}'_\infty \times X \arrow[r, "(\mathrm{id}\times\mathbb{E}_\infty)"] \arrow[d, "m\times\mathrm{id}"] & \mathbf{E}'_\infty \times X \arrow[d, "\mathbb{E}_\infty"] \\
\mathbf{E}'_\infty \times X \arrow[r, "\mathbb{E}_\infty"] & X.
\end{tikzcd}$$

Definition 5.18. *Let $(\mathscr{P}(\sigma), \phi_{\mathscr{P}(\sigma)})$ be a $\mathcal{G}$-shtuka (bounded by μ) on $X(\sigma)^{\log\diamond}$. We say that there is an $\mathbf{E}'_\infty$-action on $\mathscr{P}(\sigma)$ if there is an isomorphism*

$$\varphi(\mathscr{P}(\sigma)) : p_2^*(\mathscr{P}(\sigma), \phi_{\mathscr{P}(\sigma)}) \xrightarrow{\sim} \mathbb{E}_\infty(\sigma)^*(\mathscr{P}(\sigma), \phi_{\mathscr{P}(\sigma)})$$

on $(\mathbf{E}'_\infty \times X(\sigma))^{\log\diamond}$ that satisfies similar conditions above.

The following proposition was inspired by [Har89, 4.1.1] (cf. [HZ01, Prop. 1.3.5]).

Theorem 5.19. *Let $(\mathcal{G}, \mu)$ be as in §1.3. With the conventions above, there is a natural equivalence of categories between*

- *The category $\mathrm{Sht}^\diamond_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X)$ of $\mathcal{G}$-shtukas $(\mathscr{P}, \phi_{\mathscr{P}})$ with one leg bounded by μ on $X^\diamond$ that are equipped with equivariant $\mathbf{E}'_\infty$-actions;*
- *The category $\mathrm{Sht}^\diamond_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X(\sigma), \mathcal{M}_\sigma)$ of log $\mathcal{G}$-shtukas $(\mathscr{P}(\sigma), \phi_{\mathscr{P}(\sigma)})$ with one leg bounded by μ on $X^{\log\diamond}$ that are equipped with equivariant $\mathbf{E}'_\infty$-actions.*

The morphisms in the two categories are those preserving the $\mathbf{E}'_\infty$-actions. In fact, we show that every object in the first category can be uniquely extended to the second one.

Note that the last statement in the theorem is stronger than saying the restriction functor is essentially surjective.

We prove this theorem in §5.3.3. An immediate corollary is:

Corollary 5.20. *Let (G, X) be any Shimura datum. If the $\mathcal{P}_\Phi^*$-shtuka $\mathscr{P}_E$ determined by $\mathbb{P}^*_{K_\Phi}$ extends to a $\mathcal{P}_\Phi^*$-shtuka $\mathscr{P}$ on $(\Delta^\circ_{\Phi,K} \backslash \mathscr{S}_{K_\Phi})^\diamond$, it uniquely extends to a log shtuka $\mathscr{P}^{\mathrm{can}}$ on $(\Delta^\circ_{\Phi,K} \backslash \mathscr{S}_{K_\Phi}(\sigma))^{\log\diamond}$.*

Proof. There is a map between $\underline{\mathcal{P}_\Phi^*(\mathbb{Z}_p)}$-torsors

$$\mathbb{P}^*_{K_\Phi} \to \mathbb{P}_{K_{\Phi^*}}(P_\Phi^*, D_{\Phi^*})$$

over a map

$$\Delta^\circ_{\Phi,K} \backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi) \to \mathrm{Sh}_{K_{\Phi^*}}(P_\Phi^*, D_{\Phi^*}),$$

where Φ^* is the one defined below Definition 3.10.

The first scheme is a torus torsor under $\mathbf{E}_{\widetilde{K}_\Phi}$, which corresponds to a lattice (see [Wu25, 1.3.3])

$$\mathbf{\Lambda}_{K_\Phi} := p_2(Z(\mathbb{Q}) \times U_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1});$$

since K is neat, this lattice maps to

$$\mathbf{\Lambda}^*_{K_\Phi} := p_2((1 \times U_\Phi(\mathbb{A}_f)) \cap g_\Phi K^c g_\Phi^{-1}).$$

Thus, the map

$$\Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi) \to \mathrm{Sh}_{K_{\Phi^*}}(P^*_\Phi, D_{\Phi^*})$$

is equivariant under the isogeny between tori $\mathbf{E}_{\widetilde{K}_\Phi} \to \mathbf{E}_{K_{\Phi^*}}$.

Now we use the proof of Lemma 3.27 verbatim, replacing $\mathrm{Sh}_K(P, \mathcal{X}) \to \mathrm{Sh}_{K^c}(P^c, \mathcal{X}^c)$ with $\Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi) \to \mathrm{Sh}_{K_{\Phi^*}}(P^*_\Phi, D_{\Phi^*})$. We then see that there is an $\mathbf{E}'_{\widetilde{K}_\Phi,\infty} := \varprojlim_{x \mapsto x^p} \mathbf{E}_{\widetilde{K}_\Phi}$-action. That is, there is a diagram

$$\begin{tikzcd}
\mathbf{E}'_{\widetilde{K}_\Phi,\infty} \times \mathbb{P}^*_{K_\Phi} \arrow[r] \arrow[d] & \mathbb{P}^*_{K_\Phi} \arrow[d] \\
\mathbf{E}_{\widetilde{K}_\Phi} \times \Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi} \arrow[r] & \Delta^\circ_{\Phi,K}\backslash \mathrm{Sh}_{K_\Phi}.
\end{tikzcd}$$

Thus, by [PR24, Thm. 2.7.7] and the Tannakian formalism, $\mathscr{P}$ is equipped with an $\mathbf{E}'_{\widetilde{K}_\Phi,\infty}$-action. Note that although $\mathbf{E}'_{\widetilde{K}_\Phi,\infty} \times \mathscr{S}$ is not of finite type, the argument in *loc. cit.* still works. Indeed, denote $\mathscr{S} := \Delta^\circ_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}$. It suffices to work with an affine open $\mathrm{Spec}\, A \subset \mathscr{S}$. By the proof of *loc. cit.*, we find a perfectoid cover $\mathrm{Spa}(\widetilde{R}, \widetilde{R}^+) \to \mathrm{Spa}(\widehat{A}[1/p], \widehat{A})$ by taking the completion of finite étale extensions of $\widehat{A}[1/p]$ in its fraction field. Then we can replace $\mathrm{Spa}(\widetilde{R}, \widetilde{R}^+)$ with the perfectoid space $\mathrm{Spa}(\widetilde{R}\langle \mathsf{R} \rangle, \widetilde{R}^+\langle \mathsf{R} \rangle)$ (see [DLLZ23a, Lem. 2.2.15]), where R is the character group of $\mathbf{E}'_{\widetilde{K}_\Phi,\infty}$; and the rest of the proof in *loc. cit.* works for $\mathbf{E}'_{\widetilde{K}_\Phi,\infty} \times \mathrm{Spec}\, A$.

We then obtain $\mathscr{P}(\sigma)$ extending $\mathscr{P}$ by applying Theorem 5.19. The uniqueness follows from Corollary 2.56. ☐

5.3.3. *Proof of Theorem 5.19.* There is a natural restriction functor

$$\mathrm{Res} : \mathrm{Sht}^\diamond_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X(\sigma), \mathcal{M}_\sigma) \to \mathrm{Sht}^\diamond_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X).$$

It is fully faithful by Corollary 2.56. We show that the last statement in the theorem is true; in particular, the restriction functor is essentially surjective.

Let $\mathsf{P} := \mathsf{P}_\sigma$ and $\mathsf{Q} := \sigma^\vee \cap M$. Let $\mathsf{Q}_\infty := \mathsf{Q}[\frac{1}{p}]$. There is a natural injective homomorphism between monoids $j : \mathsf{Q} \hookrightarrow \mathsf{Q}_\infty$. Also, there is an injective homomorphism $\psi : \mathsf{Q} \to \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$ given by the embeddings $\mathsf{Q} \hookrightarrow \mathsf{Q}^{\mathrm{gp}}$ and j.

Construction 5.21. There is a natural coaction of $\mathsf{Q}^{\mathrm{gp}}_\infty$ on $\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$.

In fact, the action of $\mathbf{E}'_\infty$ on $\mathbf{E}'_\infty(\sigma)$ induces a coaction of $\mathsf{Q}^{\mathrm{gp}}_\infty$ on Q_∞. Explicitly, this is given by

$$\mathsf{Q}_\infty \to \mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}}_\infty;\ x \mapsto (x, x).$$

Similarly, the action of $\mathbf{E}'_\infty$ on $\mathbf{E}$ given by $\mathbf{E}'_\infty \times \mathbf{E} \xrightarrow{p_\infty \times \mathrm{id}} \mathbf{E} \times \mathbf{E} \xrightarrow{\text{multiplication}} \mathbf{E}$ also induces a comultiplication $\mathsf{Q}^{\mathrm{gp}} \to \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}^{\mathrm{gp}}_\infty$.

Combining the two (co)actions, we obtain a map between monoids

$$d_\infty(\sigma) : \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty \to \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}}_\infty;\ (x, y) \mapsto (x, y, -x + y).$$

This corresponds to an action $d_\infty(\sigma) : \mathbf{E}'_\infty \times \mathbf{E} \times \mathbf{E}'_\infty(\sigma) \to \mathbf{E} \times \mathbf{E}'_\infty(\sigma)$ that pointwisely sends (g, a, b) to $(g^{-1}a, gb)$. Note that the quotient of $\mathbf{E} \times \mathbf{E}'_\infty(\sigma)$ by this action is $\mathbf{E} \times^{\mathbf{E}'_\infty} \mathbf{E}'_\infty(\sigma) \cong \mathbf{E}(\sigma)$.

Denote by d_∞ the $\mathbf{E}'_\infty$-action $d_\infty(\sigma)$ restricted to $\mathbf{E} \times \mathbf{E}'_\infty(\sigma)$. ☐

We need a lemma on saturated fiber products.

Lemma 5.22. *We have that* $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{sat}} \cong \mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}}_\infty$*. Moreover, the* $\mathsf{Q}^{\mathrm{gp}}_\infty$*-coaction on the second factor* $\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$ *induces the comultiplication on the right-hand side on the second factor of* $\mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}}_\infty$*.*

Proof. We first compute $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{int}}$. Since $\mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$ is integral and $(\mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty)^{\mathrm{gp}} = \mathsf{Q}^{\mathrm{gp}}_\infty \oplus \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}^{\mathrm{gp}}_\infty$, by [Ogu18, I. Prop. 1.3.4], $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{int}}$ is the image of $\mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$ in $\mathsf{Q}^{\mathrm{gp}}_\infty \oplus_{j^{\mathrm{gp}},\mathsf{Q}^{\mathrm{gp}},\psi^{\mathrm{gp}}} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}^{\mathrm{gp}}_\infty)$. Therefore,

$$(5.13) \qquad (\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{int}} \subset (\mathsf{Q}_\infty + \mathsf{Q}^{\mathrm{gp}}) \oplus_{j^{\mathrm{gp}},\mathsf{Q}^{\mathrm{gp}},\psi^{\mathrm{gp}}} (\mathsf{Q}^{\mathrm{gp}} \oplus (\mathsf{Q}_\infty + \mathsf{Q}^{\mathrm{gp}})),$$

where $(\mathsf{Q}_\infty + \mathsf{Q}^{\mathrm{gp}})$ denotes the monoid in $\mathsf{Q}^{\mathrm{gp}}_\infty$ generated by Q_∞ and Q^{gp}.

Denote by (a, b, c) an element in $\mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$. Then $(a, b, c) \sim (a + b, 0, c - b)$ in the RHS of (5.13) since $(0, b, b) \sim (b, 0, 0)$ by amalgamated product. Hence, $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{int}}$ is generated by $(a, 0, 0)$, $(0, 0, c)$, where $a, c \in \mathsf{Q}_\infty$, and $(b, 0, -b)$, where $b \in \mathsf{Q}^{\mathrm{gp}}$.

Thus, $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{sat}} = \mathsf{Q}_\infty \oplus \mathsf{Q}^{\mathrm{gp}}_\infty$. Indeed, for any $(x, 0, y) \in (\mathsf{Q}^{\mathrm{gp}}_\infty, 0, \mathsf{Q}^{\mathrm{gp}}_\infty)$ such that $(nx, 0, ny) = (a + b, 0, c - b)$, by enlarging n, we assume that nx and ny are in Q^{gp}; and therefore, a and c are in $\mathsf{Q}^{\mathrm{gp}} \cap \mathsf{Q}_\infty = \mathsf{Q}$. We then write $(a + b, 0, c - b) = (a + b + c - c, 0, c - b) = (a + c, 0, 0) + (b - c, 0, c - b)$. Hence, y can be any element in $\mathsf{Q}^{\mathrm{gp}}_\infty$ and $x + y$ can be any element in Q_∞. Note that the generators of $(\mathsf{Q}_\infty \oplus_{j,\mathsf{Q},\psi} (\mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty))^{\mathrm{sat}}$ are $(u, 0, 0) \in (\mathsf{Q}_\infty, 0, 0)$ and $v \in \mathsf{Q}^{\mathrm{gp}}_\infty \xrightarrow{(-v,v)} (\mathsf{Q}^{\mathrm{gp}}_\infty, 0, \mathsf{Q}^{\mathrm{gp}}_\infty)$.

Now we check the last sentence. The induced $\mathsf{Q}^{\mathrm{gp}}_\infty$-coaction on the RHS of (5.13) sends $(u, v, w) \in (\mathsf{Q}_\infty + \mathsf{Q}^{\mathrm{gp}}) \oplus_{j^{\mathrm{gp}},\mathsf{Q}^{\mathrm{gp}},\psi^{\mathrm{gp}}} (\mathsf{Q}^{\mathrm{gp}} \oplus (\mathsf{Q}_\infty + \mathsf{Q}^{\mathrm{gp}}))$ to $(u, v, w, -v + w)$; note that this is compatible with the amalgamated product as the coactions on the last two factors are canceled and therefore coincide with the trivial action on the first factor. Hence, the induced coaction on $(-v, 0, v)$ is the comultiplication on the third factor, while the coaction on $(u, 0, 0)$ is trivial. ☐

We now go back to the proof. Note that

Lemma 5.23. *It suffices to show the last statement of Theorem 5.19 over an étale cover* $\widetilde{Y}$ *of* Y*.*

Proof. Let $\widetilde{X} := \widetilde{Y} \times_Y X$ and $\widetilde{X}(\sigma) := \widetilde{Y} \times_Y X(\sigma)$. Suppose that there is a log shtuka $\widetilde{\mathscr{P}}(\sigma)$ in $\mathrm{Sht}^{\diamond}_{\mathcal{G},\mu,\mathbf{E}'_\infty}(\widetilde{X}(\sigma), \mathcal{M}_{\widetilde{X}(\sigma)})$ that extends some $\widetilde{\mathscr{P}}$ in $\mathrm{Sht}^{\diamond}_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X)$ to $\widetilde{X}(\sigma)^{\log\diamond}$. Since $\widetilde{\mathscr{P}}$ is the pullback of some $\mathscr{P}$ in $\mathrm{Sht}^{\diamond}_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X)$. By Corollary 2.56, the descent datum of $\widetilde{\mathscr{P}}$ from $\widetilde{X}$ to X extends uniquely to a strict étale descent datum of $\widetilde{\mathscr{P}}(\sigma)$ from $\widetilde{X}(\sigma)$ to $X(\sigma)$, as σ is a smooth cone decomposition and the generic fiber of X is smooth. By strict étale descent of log shtukas (see Lemma 2.36), there is a log shtuka $\mathscr{P}(\sigma)$ over $(X(\sigma))^{\log\diamond}$ descending $\widetilde{\mathscr{P}}(\sigma)$. ☐

By Lemma 5.23, we can assume that X is a trivial torsor over Y, and that $X \times \mathbf{E}'_\infty(\sigma) \to X(\sigma)$ admits a chart $\mathsf{Q} \to \mathsf{Q}^{\mathrm{gp}} \oplus \mathsf{Q}_\infty$.

Let $(S^\sharp = \mathrm{Spa}(R, R^+), \mathcal{M}_{S^\sharp}, f) \in (X(\sigma), \mathcal{M}_\sigma)^{\log\diamond}(S)$, where

$$f : (S^\sharp, \mathcal{M}_{S^\sharp}) \to (S^{\sharp,+} := \operatorname{Spec} R^+, \mathcal{M}_{S^{\sharp,+}}) \xrightarrow{f^+} (X(\sigma), \mathcal{M}_\sigma).$$

We assume that $\mathcal{M}_{S^\sharp} = \mathcal{M}^{\mathrm{can}}_{S^\sharp}$ by Proposition 2.38. And f corresponds to an R^+-point of $X(\sigma)$ denoted by $f^+ : (\operatorname{Spec} R^+, \mathcal{M}_{S^{\sharp,+}}) \to (X(\sigma), \mathcal{M}_\sigma)$ that admits a chart $\mathsf{Q} \to \mathsf{Q}_\infty$ by Lemma 2.40.

Consider the fiber product in the category of saturated log schemes

$$(5.14) \qquad \begin{tikzcd} \widetilde{T}^{\sharp,+} \arrow[r] \arrow[d] & X \times \mathbf{E}'_\infty(\sigma) \arrow[d, "\alpha_\infty"] \\ \operatorname{Spec} R^+ \arrow[r] & X(\sigma). \end{tikzcd}$$

Lemma 5.24. *The completion of* $\widetilde{T}^{\sharp,+}$ *with respect to the topology of* R^+ *is the spectrum of the integral perfectoid ring* $R^+\langle \mathsf{Q}^{\mathrm{gp}}_\infty \rangle$ *equipped with an* $\mathbf{E}'_\infty$*-action.*

Proof. Recall that we can work étale locally over Y by Lemma 5.23, and assume that $X = Y \times \mathbf{E}$ and that Y is affinoid. Set $X_\infty(\sigma) := X(\sigma) \times_{\operatorname{Spec} \mathbb{Z}_p[\mathsf{Q}]} \operatorname{Spec} \mathbb{Z}_p[\mathsf{Q}_\infty]$. Then

$$\begin{aligned}
&(S^{\sharp,+}, \mathcal{M}_{S^{\sharp,+}}) \times^{\mathrm{sat}}_{X(\sigma)} (X \times \mathbf{E}'_\infty(\sigma)) \\
&= (\operatorname{Spec} R^+, \mathcal{M}_{S^{\sharp,+}}) \times_{X_\infty(\sigma)} X_\infty(\sigma) \times^{\mathrm{sat}}_{X(\sigma)} (X \times \mathbf{E}'_\infty(\sigma)) \\
&= (\operatorname{Spec} R^+, \mathcal{M}_{S^{\sharp,+}}) \times_{X_\infty(\sigma)} X_\infty(\sigma)[\mathsf{Q}^{\mathrm{gp}}_\infty].
\end{aligned}$$

The second line follows from Lemma 2.40 and the third line follows from Lemma 5.22. ☐

We now finish the proof. We omit $\phi_{\mathscr{P}}$ in the proof. Suppose that we are given $\mathscr{P}$ in $\mathrm{Sht}^\diamond_{\mathcal{G},\mu,\mathbf{E}'_\infty}(X)$. Set $\widetilde{T}^\sharp := \operatorname{Spa}(R\langle \mathsf{Q}^{\mathrm{gp}}_\infty\rangle, R^+\langle \mathsf{Q}^{\mathrm{gp}}_\infty\rangle)$, which is affinoid perfectoid by [DLLZ23a, Lem. 2.2.5].

Note that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{E}'_\infty \times \mathbf{E}'_\infty \times X \arrow[r, shift left, "d'_\infty := d_\infty \circ (i \times \mathrm{id} \times \mathrm{id})"] \arrow[r, shift right, "p_{23}"'] \arrow[d, "\mathrm{id} \times \varphi_\infty"] & \mathbf{E}'_\infty \times X \arrow[r, "\mathbb{E}_\infty"] \arrow[d, "\varphi_\infty"] & X \arrow[d, equal] \\
\mathbf{E}'_\infty \times \mathbf{E}'_\infty \times X \arrow[r, shift left, "m \times \mathrm{id}"] \arrow[r, shift right, "p_{23}"'] & \mathbf{E}'_\infty \times X \arrow[r, "p_2"] & X.
\end{tikzcd} \tag{5.15}$$

Here, d_∞ also denotes the action of the first factor $\mathbf{E}'_\infty$ on $\mathbf{E}'_\infty \times X$ that is a (positive) multiplication on the first factor $\mathbf{E}'_\infty$ and $-\mathbb{E}_\infty$ on the second factor X; m (resp. i) denotes the multiplication (resp. inverse) on $\mathbf{E}'_\infty$; d'_∞ maps (a, b, x) to $(a^{-1}b, ax)$.

Let $\mathscr{P}_2$ be the pullback of $\mathscr{P}$ to $(\mathbf{E}'_\infty \times X)^\diamond$ along the projection p_2 to the second factor. Note that $p_2 \circ d'_\infty = p_2 \circ \varphi_\infty \circ p_{13} = \mathbb{E}_\infty \circ p_{13}$. From this, we obtain a descent datum $\varphi(\mathscr{P}_2) : p^*_{23}\mathscr{P}_2 \xrightarrow{\sim} d'^*_\infty \mathscr{P}_2$ by pulling back $\varphi(\mathscr{P}) : p^*_2\mathscr{P} \xrightarrow{\sim} \varphi^*_\infty p^*_2 \mathscr{P} = \mathbb{E}^*_\infty \mathscr{P}$ along p_{13}. The cocycle condition is given by the group action $\varphi(\mathscr{P})$. Therefore, $\mathscr{P}_2$ descends to $\mathscr{P}'$ on X with this descent datum.

On the other hand, denote by $\mathscr{P}_3 := \mathbb{E}^*_\infty \mathscr{P} = \varphi^*_\infty p^*_2 \mathscr{P}$, we have a standard descent datum $\varphi(\mathscr{P}_3) : p^*_{23}\mathscr{P}_3 \xrightarrow{\sim} d'^*_\infty \mathscr{P}_3$ given by the pullback of $\mathscr{P}$ along $\mathbb{E}_\infty$. Note that $\varphi(\mathscr{P})$ gives an isomorphism between descent data $(\mathscr{P}_2, \varphi(\mathscr{P}_2)) \to (\mathscr{P}_3, \varphi(\mathscr{P}_3))$, and it descends to an isomorphism $\mathscr{P}' \xrightarrow{\sim} \mathscr{P}$. Indeed, define $\boldsymbol{\alpha} : \mathbf{E}'_\infty \times \mathbf{E}'_\infty \times X \xrightarrow{\sim} \mathbf{E}'_\infty \times \mathbf{E}'_\infty \times X$ by the assignment $(a, b, x) \mapsto (a^{-1}b, a, x)$. Then $p_{23} \circ \boldsymbol{\alpha} = p_{13}$, $(\mathrm{id} \times \mathbb{E}_\infty) \circ \boldsymbol{\alpha} = d'_\infty$ and $(m \times \mathrm{id}) \circ \boldsymbol{\alpha} = p_{23}$. Pulling back (5.12) along $\boldsymbol{\alpha}$, we obtain a commutative diagram

$$\begin{tikzcd}
p^*_{13}p^*_2\mathscr{P} \arrow[r, "p^*_{13}\varphi(\mathscr{P})"] \arrow[d, equal] & p^*_{13}\mathbb{E}^*_\infty\mathscr{P} = d^{\prime,*}_\infty p^*_2\mathscr{P} \arrow[r, "d^{\prime,*}_\infty\varphi(\mathscr{P})"] & d^{\prime,*}_\infty\mathbb{E}^*_\infty\mathscr{P} \arrow[d, equal] \\
p^*_{23}p^*_2\mathscr{P} \arrow[rr, "p^*_{23}\varphi(\mathscr{P})"] & & p^*_{23}\mathbb{E}^*_\infty\mathscr{P}.
\end{tikzcd} \tag{5.16}$$

We write (5.16) above in the following form:

$$\begin{tikzcd}
p^*_3\mathscr{P} \arrow[r, "p^*_{13}\varphi(\mathscr{P})"] \arrow[d, "p^*_{23}\varphi(\mathscr{P})"] & p^*_{13}\mathbb{E}^*_\infty\mathscr{P} = d^{\prime,*}_\infty p^*_2\mathscr{P} \arrow[d, "d^{\prime,*}_\infty\varphi(\mathscr{P})"] \\
p^*_{23}\mathbb{E}^*_\infty\mathscr{P} = (\mathrm{id} \times \varphi_\infty)^* p^*_3\mathscr{P} \arrow[r, equal] & (\mathrm{id} \times \varphi_\infty)^*(m \times \mathrm{id})^* p^*_2\mathscr{P} = d^{\prime,*}_\infty\mathbb{E}^*_\infty\mathscr{P};
\end{tikzcd}$$

this implies the claim that $\varphi(\mathscr{P})$ is an isomorphism between the descent data of $\mathscr{P}_2$ and $\mathscr{P}_3$.

Moreover, we show that $\mathscr{P}' \xrightarrow{\sim} \mathscr{P}$ is an automorphism of $\mathscr{P}$. Consider the canonical section $s : X \to \mathbf{E}'_\infty \times X$ sending x to (e, x), it is a section of both $\mathbb{E}_\infty$ and p_2. Then,

$$\mathscr{P}' = s^*\mathbb{E}^*_\infty\mathscr{P}' = s^*\mathscr{P}_2 = s^*p^*_2\mathscr{P} = \mathscr{P}.$$

Also, the descended automorphism $\mathscr{P} \xrightarrow{\sim} \mathscr{P}$ is an identity, as it can be realized as the pullback $s^*\varphi(\mathscr{P}) : \mathscr{P} \xrightarrow{\sim} \mathscr{P}$ of the morphism $\varphi(\mathscr{P})$ between the descent data; here we use the fact that the following composition is the identity:

$$X \xrightarrow{s} \mathbf{E}'_\infty \times X \overset{\varphi_\infty}{\to} \mathbf{E}'_\infty \times X \overset{p_2}{\to} X.$$

This automorphism is an identity by the first condition in Definition 5.17.

Next, we extend the first row of (5.15) to

$$\begin{tikzcd}[ampersand replacement=\&] \mathbf{E}'_\infty \times \mathbf{E}'_\infty(\sigma) \times X \arrow[r, shift left, "d'_\infty(\sigma):=d_\infty(\sigma)\circ(i\times\mathrm{id}\times\mathrm{id})"] \arrow[r, shift right, "p_{23}"'] \& \mathbf{E}'_\infty(\sigma) \times X \arrow[r, "\mathbb{E}_\infty(\sigma)'"] \& X(\sigma). \end{tikzcd} \tag{5.17}$$

Here, $\mathbb{E}_\infty(\sigma)'$ is defined by $\mathbf{E}'_\infty(\sigma) \times X \to \mathbf{E}(\sigma) \times X \to \mathbf{E}(\sigma) \times^{\mathbf{E}} X = X(\sigma)$ and it extends the map $\mathbb{E}_\infty$. Again, we equip $\mathbf{E}'_\infty(\sigma)$ with the positive multiplication of $\mathbf{E}'_\infty$ and X with the negative action. Thus, $d_\infty(\sigma)$ extends d_∞ (resp. $d'_\infty(\sigma)$ extends d'_∞).

Denote by $\mathscr{Q}$ the pullback of $\mathscr{P}$ to $(\mathbf{E}'_\infty(\sigma) \times X)^{\log\diamond}$ via the projection to the second factor X.

For any $f^+ : (\operatorname{Spec} R^+, \mathcal{M}_{R^+}) \to X(\sigma)$ as above, the evaluation $\mathscr{Q}_{\widetilde{T}^\sharp}$ of $\mathscr{Q}$ at $\widetilde{T}^\sharp \to \widetilde{T}^{\sharp,+} \to X \times \mathbf{E}'_\infty(\sigma)$ is equipped with an $\mathbf{E}'_\infty$-action given by the pullback of $d'_\infty(\sigma)$ by construction and by Lemma 5.24. By v-descent of shtukas (see [SW20, Prop. 19.5.3]), the shtuka $\mathscr{Q}_{\widetilde{T}^\sharp}$ descends to a shtuka $\mathscr{Q}_{S^\sharp}$.

This assignment can be easily checked to be functorial, and we denote the obtained shtuka on $X(\sigma)^{\log\diamond}$ by $\mathscr{P}(\sigma)$; the $\mathbf{E}'_\infty$-action on $\mathscr{P}(\sigma)$ is descended from that of $\mathscr{Q}$ by the multiplication action of $\mathbf{E}'_\infty$ on $\mathbf{E}'_\infty(\sigma)$. Moreover, when we restrict this construction to $(\mathbf{E}'_\infty \times X)^\diamond$, we have seen that we obtain a shtuka $\mathscr{P}' = \mathscr{P}$. Now we have completed the proof of last statement in Theorem 5.19. □

5.4. **Canonical integral models of compactifications.**

5.4.1. *Assumptions.* As mentioned in Remark 5.6, the extension of shtukas to toroidal compactifications $\mathscr{S}_K^\Sigma$ can be achieved without understanding the geometry of the interior $\mathscr{S}_K$. To highlight this point, we present a sequence of assumptions and definitions at slightly different levels of generality.

Let (G, X) be a Shimura datum, let $\mathcal{G}$ be a quasi-parahoric model of $G = G_{\mathbb{Q}_p}$, and let $K_p = \mathcal{G}(\mathbb{Z}_p)$. Let $\{\mathscr{S}_{K_pK^p}(G, X)\}_{K^p \subset G(\mathbb{A}_f^p)}$ be a family of integral models of $\{\mathrm{Sh}_{K_pK^p}(G, X)\}_{K^p \subset G(\mathbb{A}_f^p)}$.

To study the boundary stratification of the integral model of the toroidal compactification and to construct log shtukas on it, we assume:

Assumption 5.25. *For each neat* $K^p \subset G(\mathbb{A}_f^p)$,

1. $\mathscr{S}_K(G, X)$ *has a good compactification theory as in Axiom 5.1. (In fact, we only need the assumptions for toroidal compactifications.)*
2. *For any* $\Phi \in \mathcal{CLR}(G, X)$, *the* $\mathcal{P}_\Phi^*$*-shtuka* $(\mathscr{P}_{\Phi,E}, \phi_{\mathscr{P}_{\Phi,E}})$ *over* $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\Diamond$ *with one leg bounded by* μ_Φ^* *associated with the push-out of the de Rham pro-étale* $\underline{\mathcal{P}_\Phi^c(\mathbb{Z}_p)}$*-torsor* $\mathbb{P}_{K_\Phi} \to \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$ *via* $\mathcal{P}_\Phi^c \to \mathcal{P}_\Phi^*$ *extends to a* $\mathcal{P}_\Phi^*$*-shtuka* $(\mathscr{P}_\Phi, \phi_{\mathscr{P}_\Phi})$ *over* $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^{\Diamond/}$.

To consider the boundary stratification on the integral model of the minimal compactification and prove the well-positionedness of various strata on the special fiber, we further assume:

Assumption 5.26. *For each neat* $K^p \subset G(\mathbb{A}_f^p)$, *assumptions 5.25 hold. Moreover, for each* $[\Phi] \in \mathcal{CLR}_K(G, X)$, $\Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{\overline{K}_\Phi}(\overline{P}_\Phi, \overline{D}_\Phi) \to \Delta_{\Phi,K}^\circ \backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ *is an abelian scheme torsor.*

To consider functoriality and uniqueness of integral models of toroidal (resp. minimal compactifications), we make the following definition:

Definition 5.27. *Fix $K' = K_pK'^{,p}$ for K_p a quasi-parahoric subgroup and $K'^{,p}$ neat open compact. Assume $\mathscr{S}_{K'} := \mathscr{S}_{K'}(G, X)$ has a good compactification theory as in Axiom 5.1 and let Σ be a projective smooth cone decomposition for (G, X, K').*

*We say that $\{\mathscr{S}_K^{\Sigma}\}_{K^p}$ (resp. $\{\mathscr{S}_K^{\min}\}_{K^p}$) is a system of **canonical integral models** (in the sense of Pappas-Rapoport) of toroidal compactifications $\{\mathrm{Sh}_K^{\Sigma}\}_{K^p}$ (resp. minimal compactifications $\{\mathrm{Sh}_K^{\min}\}_{K^p}$) if, for any $\Phi \in \mathcal{CLR}(G, X)$, the integral model of mixed Shimura variety $\{\mathscr{S}_{K_\Phi}\}_{K^p}$ (resp. $\{\mathscr{S}_{K_{\Phi,h}}\}_{K^p}$) is a system of canonical integral models in the sense of Axiom 4.2 (resp. Axiom 4.1). The inverse system runs over a cofinal collection of neat open compact subgroups $K^p \subset K'^{,p}$ with the cone decomposition the induced one (see* [Wu25, Def. 1.18(2)]*).*

Theorem 5.28. *Let (G, X) be any abelian-type Shimura datum, let $p > 0$ be any prime, and let $\mathcal{G}$ be any quasi-parahoric model with $K_p = \mathcal{G}(\mathbb{Z}_p)$. Then $\{\mathrm{Sh}_{K_pK^p}(G, X)\}_{K^p \subset G(\mathbb{A}_f^p)}$ has a* canonical integral model *$\{\mathscr{S}_{K_pK^p}\}_{K^p \subset G(\mathbb{A}_f^p)}$ in the sense of Axiom 4.1 that satisfies Assumption 5.26. Moreover, $\{\mathscr{S}_{K_pK^p}\}_{K^p \subset G(\mathbb{A}_f^p)}$ has canonical integral models of toroidal compactifications and minimal compactifications for any given neat open compact $K'^{,p}$.*

Proof. In Theorem 6.26, we will check Assumption 5.25(2) and the second assertion using the construction in [Wu25] (specializing to Case (STB_1) and its normalization models with corresponding quasi-parahoric levels; see *loc. cit.* Sec. 4.2). Assumption 5.25(1) (i.e., Axiom 5.1) is Theorem 5.5. The second statement of Assumption 5.26 is [Wu25, Prop. 4.59]; as one will see later, this, in fact, requires Proposition 6.21. ☐

5.4.2. We mention some important consequences. The following proposition means that the shtukas on the strata *automatically* glue together.

Proposition 5.29. *Under Assumption 5.25, there is a unique morphism*

$$(\mathscr{S}_K^{\Sigma})^{\log\Diamond} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$$

extending $\mathscr{S}_K^{\Diamond/} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c}$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
(\mathscr{S}_K^{\Sigma})^{\log\Diamond} \arrow[d] & W^{\log\Diamond/} \arrow[l] \arrow[r] & (\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(\sigma))^{\log\Diamond/} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1} & & \mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1}. \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"']
\end{tikzcd} \tag{5.18}$$

By Lemma 2.31, there is no difference of using $\log\Diamond$ and $\log\Diamond/$ on the top-left corner of (5.18) since $\mathscr{S}_K^{\Sigma}$ is proper.

Proof. Let $\mathscr{S}_K$ be an integral model satisfying the Assumption 5.25. By Corollary 5.13 and Theorem 2.55, there is a unique $\mathcal{P}^*_\Phi$-shtuka $\mathscr{P}^*_\Phi$ on $\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}^{\Diamond/}$ such that the commutative diagram (5.10) extends integrally. Then the first sentence follows from applying Corollary 5.20 and Lemma 5.15 in order. The diagram (5.18) follows from the proof of Lemma 5.15. ☐

Combing Proposition 5.29 and Corollary 2.58, we see that

Corollary 5.30. *The log shtuka in Proposition 5.29 induces a non-log shtuka on the special fiber*

$$\mathscr{S}^{\Sigma,\Diamond}_{K,\mathcal{O}_E/\mathfrak{p}_v} = \mathscr{S}^{\Sigma,\diamond}_{K,\mathcal{O}_E/\mathfrak{p}_v} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$$

with a commutative diagram

(5.19)
$$\begin{tikzcd}
(\mathscr{S}^{\Sigma}_{K,\mathcal{O}_E/\mathfrak{p}_v})^{perf} \arrow[d] & W^{perf}_{\mathcal{O}_E/\mathfrak{p}_v} \arrow[l] \arrow[r] & (\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(\sigma))^{perf}_{\mathcal{O}_E/\mathfrak{p}_v} \arrow[d] \\
\mathrm{Sht}^{W}_{\mathcal{G}^c,\mu^c,\delta=1} & & \mathrm{Sht}^{W}_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1}. \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"']
\end{tikzcd}$$

Corollary 5.31. *Under Assumption 5.25, for any* $\Upsilon = [(\Phi,\sigma)] \in \mathrm{Cusp}_K(G,X,\Sigma)$*, there is a map*

$$(\mathcal{Z}_{\Upsilon,K})^{\log ?} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1},$$

where $? = \lozenge, \diamond$ *or* $\lozenge/$ *and the log structure on* $\mathcal{Z}_{\Upsilon,K}$ *is defined by pulling back the log structure on* $\mathscr{S}^{\Sigma}_K$*. In particular, there is a morphism*

$$\mathscr{S}^{\lozenge}_K \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$$

extending $\mathscr{S}^{\lozenge/}_K \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$.

Proof. Since $\mathscr{S}^{\Sigma}_K$ is proper, $(\mathscr{S}^{\Sigma}_K)^{\log\lozenge} = (\mathscr{S}^{\Sigma}_K)^{\log\lozenge/} = (\mathscr{S}^{\Sigma}_K)^{\log\diamond}$ (see Lemma 2.31). For a map between fs log schemes $(X,\mathcal{M}_X) \to (Y,\mathcal{M}_Y)$ that are separated and of finite type, by definition, there is a map between log diamonds $(X,\mathcal{M}_X)^{\log ?} \to (Y,\mathcal{M}_Y)^{\log ?}$. For the last sentence, note that the pullback log structure to the interior is the trivial log structure. Then it follows from Corollary 2.39. ☐

5.4.3.

Lemma 5.32. *Under Assumption 5.26, for any* $\Phi \in \mathcal{CLR}(G,X)$*, the* $\overline{\mathcal{P}}^*_\Phi$*-shtuka* $(\overline{\mathscr{P}}_{\Phi,E}, \phi_{\overline{\mathscr{P}}_{\Phi,E}})$ *over* $\mathrm{Sh}^{\lozenge}_{\overline{K}_\Phi}$ *with one leg bounded by* $\overline{\mu}^*_\Phi$ *associated with the push-out of the de Rham pro-étale* $\underline{\overline{\mathcal{P}}^c_\Phi(\mathbb{Z}_p)}$*-torsor* $\overline{\mathbb{P}}_{K_\Phi} \to \mathrm{Sh}_{\overline{K}_\Phi}$ *via* $\overline{\mathcal{P}}^c_\Phi \to \overline{\mathcal{P}}^*_\Phi$ *extends to a* $\overline{\mathcal{P}}^*_\Phi$*-shtuka* $(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ *over* $\mathscr{S}^{\lozenge/}_{\overline{K}_\Phi}$*. In other words, we have a morphism* $(\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{\overline{K}_\Phi})^{\lozenge/} \to \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}$ *extending the one on generic fiber. Similarly, we have a morphism* $(\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}})^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h}}$ *extending the one on generic fiber.*

Proof. Zariski locally over $\mathscr{S}_{\overline{K}_\Phi} =: C(\Phi)$, $\mathscr{S}_{K_\Phi} \to \mathscr{S}_{\overline{K}_\Phi}$ is the projection $\mathbf{E}(\Phi) \times_{\mathrm{Spec}\,\mathbb{Z}} C(\Phi) \to C(\Phi)$. It admits the zero section $C(\Phi) \to \mathbf{E}(\Phi) \times C(\Phi)$. We have the following commutative diagrams:

$$\begin{tikzcd}
(\mathbf{E}(\Phi)_\eta \times C(\Phi)_\eta)^{\lozenge} \arrow[r] \arrow[d, shift left] & \mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi} \arrow[d] & (\mathbf{E}(\Phi) \times C(\Phi))^{\lozenge/} \arrow[r] \arrow[d, shift left] & \mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi} \arrow[d] \\
C(\Phi)^{\lozenge}_\eta \arrow[r] \arrow[u, shift left] & \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}, & C(\Phi)^{\lozenge/} \arrow[r, dashed] \arrow[u, shift left] & \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}.
\end{tikzcd}$$

The composition $C(\Phi)^{\lozenge/} \to \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}$ exists Zariski locally on $C(\Phi)$. Since, over generic fiber, we have a globally defined morphism $C(\Phi)^{\lozenge}_\eta \to \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}$, and the commutativity of the left square implies that the morphism is independent of the choice of (Zariski local) section $C(\Phi) \to \mathbf{E}(\Phi) \times C(\Phi)$, we obtain a globally defined morphism $C(\Phi)^{\lozenge/} \to \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi}$ using [PR24, Cor. 2.7.10]. This further descends to $(\Delta^{\circ}_{\Phi,K}\backslash\mathscr{S}_{\overline{K}_\Phi})^{\lozenge/}$ using the descent data coming from the generic fiber. We can apply this argument to $\mathscr{S}_{K_{\Phi,h}}$ since $\mathscr{S}_{\overline{K}_\Phi} \to \mathscr{S}_{K_{\Phi,h}}$ is an abelian scheme torsor by assumption. ☐

In particular, we have

Proposition 5.33. *Under Assumption 5.26, the commutative diagram (5.9) in Lemma 5.10 extends over integral base. That is to say, for any $[\Phi]$, we have the following commutative diagram*

$$\begin{tikzcd}
\Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^{\lozenge/} \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{\overline{K}_\Phi}(\overline{P}_\Phi, \overline{D}_\Phi)^{\lozenge/} \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^{\lozenge/} \arrow[d] \\
\mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1} \arrow[r] & \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi,\bar{\mu}^*_\Phi,\delta=1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1}.
\end{tikzcd}$$

Proposition 5.34. *Under Assumption 5.26, the commutative diagram (5.10) in Corollary 5.13 extends over the integral base. That is to say, for all such coverings W^0 (where $\mathfrak{W}$ are open affine coverings of $\mathfrak{X}^{\circ}_{[\Phi,\sigma],K}$, here for each Φ, we let σ run over all $\Sigma(\Phi)^+$), we have following commutative diagram*

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^{\lozenge} \arrow[d] & W^{0,\lozenge} \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^{\lozenge} \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^{\lozenge} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1} & & \mathrm{Sht}_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & \mathrm{Sht}_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1}.
\end{tikzcd} \tag{5.20}$$

Taking the special fibers (apply the reduction functor), we have

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^{perf}_{\bar{s}} \arrow[d] & W^{0,perf}_{\bar{s}} \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^{perf}_{\bar{s}} \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^{perf}_{\bar{s}} \arrow[d] \\
\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1} & & \mathrm{Sht}^W_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & \mathrm{Sht}^W_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1}.
\end{tikzcd} \tag{5.21}$$

Note that W^0 is not of finite type over $\mathbb{Z}_p$; nevertheless, since W^0 is affine, noetherian and flat over $\mathbb{Z}_p$, one can still apply [PR24, Cor. 2.7.10] (see the first paragraph of the proof of Theorem 2.55).

5.4.4. The compactifications satisfying Definition 5.27 are fully functorial.

Proposition 5.35. *Let $f : (G_1, X_1, \mathcal{G}_1, K_1^p, \Sigma_1) \to (G_2, X_2, \mathcal{G}_2, K_2^p, \Sigma_2)$ be a morphism. Here, the meaning of a morphism is: (1) $f : (G_1, X_1) \to (G_2, X_2)$ is a morphism between Shimura data; (2) $\mathcal{G}_1 \to \mathcal{G}_2$ is a morphism between quasi-parahoric group schemes whose generic fiber is $f : G_1 \to G_2$; (3) $K_1^p \to K_2^p$ is a morphism between neat open compact subgroups compatible with $f : G_1(\mathbb{A}_f^p) \to G_2(\mathbb{A}_f^p)$; (4) Σ_1 and Σ_2 are smooth projective and are compatible in the sense of* [Wu25, Def. 1.18(3)].

Let $\mathscr{S}^{\Sigma_i}_{K_i}(G_i, X_i)$ be a canonical integral model of $\mathrm{Sh}^{\Sigma_i}_{K_i}(G_i, X_i)$ for $i = 1, 2$. Then there is a unique morphism $f_{\Sigma_1,\Sigma_2} : \mathscr{S}^{\Sigma_1}_{K_1} \to \mathscr{S}^{\Sigma_2}_{K_2}$ extending the one $f : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ over the generic fiber by functoriality between Shimura data, and makes the following diagram commutes:

$$\begin{tikzcd}
(\mathscr{S}^{\Sigma_1}_{K_1})^{\log\lozenge} \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{G}^c_1,\mu^c_1,\delta=1} \arrow[d] \\
(\mathscr{S}^{\Sigma_2}_{K_2})^{\log\lozenge} \arrow[r] & \mathrm{Sht}_{\mathcal{G}^c_2,\mu^c_2,\delta=1}.
\end{tikzcd}$$

Proof. By Proposition 4.10, for any cusp label representative Φ_1 mapping to Φ_2, there is a morphism $f_{\Phi_1} : \mathscr{S}_{K_{\Phi_1}} \to \mathscr{S}_{K_{\Phi_2}}$ extending the one on the generic fiber. Note that this morphism is automatically a morphism between torus torsors equivariant under $(\mathbf{E}_{K_{\Phi_1}} \to \mathbf{E}_{K_{\Phi_2}})$ by separatedness and normality. Hence, we obtain a morphism of integral models $f_{\Phi_1}(\sigma_1) : \mathscr{S}_{K_{\Phi_1}}(\sigma_1) \to \mathscr{S}_{K_{\Phi_2}}(\sigma_2)$

for σ_1 mapping to σ_2; we also have a morphism between closed strata f_{Φ_1,σ_1}. Axiom (5.1.4) requires that $\Delta^\circ_{\Phi,K}$ acts on $\mathscr{S}_{K_\Phi}$. By separatedness and normality again, there are morphisms $f^\circ_{\Phi_1} : \Delta^\circ_{\Phi_1,K_1}\backslash\mathscr{S}_{K_{\Phi_1}} \to \Delta^\circ_{\Phi_2,K_2}\backslash\mathscr{S}_{K_{\Phi_2}}$, $f^\circ_{\Phi_1}(\sigma_1) : \Delta^\circ_{\Phi_1,K_1}\backslash\mathscr{S}_{K_{\Phi_1}}(\sigma_1) \to \Delta^\circ_{\Phi_2,K_2}\backslash\mathscr{S}_{K_{\Phi_2}}(\sigma_2)$ and also a morphism between their corresponding closed strata. We have the desired morphism between compactifications by [MP19, Lem. A.3.4]. The commutativity of the diagram is guaranteed by Theorem 2.55, and by the commutativity of the diagram on the generic fiber. □

In particular,

Corollary 5.36. *Fix (G, X, Σ). The inverse system of canonical integral models $\{\mathscr{S}^\Sigma_K\}_{K^p}$ is unique up to a unique isomorphism.*

Upon restricting the maps to the open dense strata, we obtain that

Corollary 5.37. *Fix (G, X). Let $\mathcal{G}_1 \to \mathcal{G}_2$ be a morphism of quasi-parahoric group schemes, then there exists a* proper *morphism $\mathscr{S}_{K_1} \to \mathscr{S}_{K_2}$ extending $\mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$.*

Similarly, one has canonicity and functoriality of canonical integral models of minimal compactifications.

Proposition 5.38. *Let $f : (G_1, X_1, \mathcal{G}_1, K^p_1) \to (G_2, X_2, \mathcal{G}_2, K^p_2)$ be a morphism as in Proposition 5.35. Let $\mathscr{S}^{\min}_{K_i}(G_i, X_i)$ be a canonical integral model of $\mathrm{Sh}^{\min}_{K_i}(G_i, X_i)$ for $i = 1, 2$. Then there is a unique morphism $f^{\min} : \mathscr{S}^{\min}_{K_1} \to \mathscr{S}^{\min}_{K_2}$ extending $f : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ over the generic fiber between Shimura varieties induced by functoriality between Shimura data.*

In particular, the inverse system of canonical integral models of minimal compactifications $\{\mathscr{S}^{\min}_K\}_{K^p}$ is unique up to a unique isomorphism.

Proof. The argument as in [MP19, Lem. A.3.4] still works. See [Wu25, Lem. 5.5] and [Mao25a, 4.1.8-4.1.9]. □

6. SHTUKAS ON INTEGRAL MODELS OF BOUNDARY MIXED SHIMURA VARIETIES OF ABELIAN TYPE

In this section, we construct shtukas on integral models of boundary mixed Shimura varieties of abelian type, and verify the axioms 4.2.

6.1. **Hodge-type case.**

6.1.1. *Set up.* Let (G, X) be a Hodge-type Shimura datum, and let $\mathcal{G}$ be a stabilizer quasi-parahoric model of G, with $K_p = \mathcal{G}(\mathbb{Z}_p)$. We fix an adjusted Hodge embedding $(G, X, K_p) \hookrightarrow (G^\ddagger, X^\ddagger, K^\ddagger_p)$ in the sense of [Mao25a, Def. 3.16, Rmk. 3.17], where $G^\ddagger = \mathrm{GSp}(V, \psi)$, $\mathcal{GSP} = \mathrm{GSp}(V_{\mathbb{Z}_p})$ is a hyperspecial model of $G^\ddagger$, $K^\ddagger_p = \mathcal{GSP}(\mathbb{Z}_p)$ is the stabilizer group of the self-dual lattice $V_{\mathbb{Z}_p}$, and $K_p = G(\mathbb{Q}_p) \cap K^\ddagger_p$. Explicitly, we work with the construction in [KPZ24], combine the chain of lattices into a single lattice and then apply the Zarhin's trick to get a self-dual lattice in the Siegel side. We fix such an adjusted Hodge embedding instead of a random one as in [PR24] in order to have a good theory of compactifications as in [MP19] and to ensure good compatibility of levels at the boundary; see [Mao25a, Prop. 1.2]. The induced morphism $\mathcal{G} \to \mathcal{GSP}$ factors through a smoothening map $\mathcal{G} \to \overline{\mathcal{G}}$, and $\overline{\mathcal{G}} \hookrightarrow \mathcal{GSP}$ is a closed embedding. Let $(s_\alpha) \in V^\otimes_{\mathbb{Z}_p}$ be the Hodge tensors that cut out the closed subscheme $\overline{\mathcal{G}}$. Note that s_α can be defined in $V^\otimes_{\mathbb{Z}_{(p)}}$.

Remark 6.1. *In the Hodge-type case, the center $Z(G)^\circ$ is an almost product of a split $\mathbb{Q}$-torus and a compact-type $\mathbb{Q}$-torus; thus $Z(G)_{ac}$ is trivial, $G = G^c$, $P_\Phi = P^c_\Phi = P^*_\Phi$, and $\Delta^\circ_{\Phi,K}$ is trivial. See* [MP19, Lem. 2.1.20].

On the boundary, since the level group $K_{\Phi,G} = gKg^{-1}$ is twisted by $g = g_\Phi \in G(\mathbb{A}_f)$, we need to twist the Hodge tensors. Let $(s_{\alpha,\Phi}) \in V^{g,\otimes}_{\mathbb{Z}_{(p)}}$ (where $V^{g}_{\mathbb{Z}_{(p)}} = gV_{\mathbb{Z}_p} \cap V_{\mathbb{Q}}$) be the collection of Hodge tensors that cut out the closed subgroup scheme $\overline{\mathcal{G}}_\Phi \to \mathcal{GSP}_{\Phi^\ddagger}$; here $\overline{\mathcal{G}}_\Phi$ and $\mathcal{GSP}_{\Phi^\ddagger}$ are the left g-conjugates of $\overline{\mathcal{G}}$ and $\mathcal{GSP}$, respectively. Indeed, we take g_p-conjugation, where g_p is the p-component of g; we do not distinguish g and g_p when the difference is clear from the context. Let $\mathcal{G}_\Phi \to \overline{\mathcal{G}}_\Phi$ be the dilated morphism; $\mathcal{G}_\Phi$ is a smooth affine model of G.

Over $\mathbb{Z}_p$, (s_α) and $(s_{\alpha,\Phi})$ are related as follows:

$$\begin{tikzcd}
V_{\mathbb{Z}_p}^{\otimes n} \arrow[r, "g"] \arrow[d, "s_\alpha"] & V_{\mathbb{Z}_p}^{g,\otimes n} \arrow[d, "s_{\alpha,\Phi}"] \\
V_{\mathbb{Z}_p}^{\otimes m} \arrow[r, "g"] & V_{\mathbb{Z}_p}^{g,\otimes m}.
\end{tikzcd}$$

6.1.2. *Reduction.* Recall that over $\mathrm{Sh}_K(G,X)$, the family of Hodge tensors $(s_\alpha) \in V^{\otimes}_{\mathbb{Z}_p}$ defines a family of étale tensors $(t_{\alpha,\text{ét}})$ on the Tate module $T_p(A)$, where A is the universal abelian scheme over $\mathrm{Sh}_K(G,X)$, pulled back from $\mathrm{Sh}_{K^\ddagger}(G^\ddagger,X^\ddagger)$. We denote by $\mathbb{L}_{\rho,V_{\mathbb{Z}_p}}$ the local system given by $T_p(A)$, where $\rho : G \to \mathrm{GSp}(V) \to \mathrm{GL}(V)$. Let $\mathbb{P}_K$ be the pro-étale torsor under $\underline{K_p}$ defined by

$$\underline{\mathrm{Isom}}_{(t_{\alpha,\text{ét}}),(s_\alpha)}(\mathbb{L}_{\rho,V_{\mathbb{Z}_p}}, V_{\mathbb{Z}_p,\mathrm{Sh}_K(G,X)}), \tag{6.1}$$

this coincides with the pro-étale torsor (3.5).

One can repeat the above process with A replaced by the universal 1-motive $\mathcal{Q}$ over $\mathrm{Sh}_{K_\Phi} := \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)$, pulled back from $\mathrm{Sh}_{K_{\Phi^\ddagger}} := \mathrm{Sh}_{K_{\Phi^\ddagger}}(P_{\Phi^\ddagger}, D_{\Phi^\ddagger})$. Here $\Phi^\ddagger \in \mathcal{CLR}(G^\ddagger, X^\ddagger)$ is the cusp-label representative induced by Φ.

By the work of Brylinski, all Hodge classes on 1-motives over $\mathbb{C}$ are absolute Hodge cycles; thus $(s_{\alpha,\Phi})$ induces $(t_{\alpha,\Phi,\text{ét}}) \in T_p(\mathcal{Q})^\otimes$ over Sh_{K_Φ}. We denote by $\mathbb{L}_{\rho_\Phi,K_\Phi}$ the local system given by the Tate module $T_p(\mathcal{Q})$. Consider the following torsor under $\underline{\mathcal{G}_\Phi(\mathbb{Z}_p)}$ over Sh_{K_Φ}:

$$\mathbb{P}_{K_{\Phi,G}} := \underline{\mathrm{Isom}}_{(t_{\alpha,\Phi,\text{ét}}),(s_{\alpha,\Phi})}(\mathbb{L}_{\rho_\Phi,K_\Phi}, V^g_{\mathbb{Z}_p,\mathrm{Sh}_{K_\Phi}}). \tag{6.2}$$

Consider the following torsor under $\underline{K_{\Phi,Q,p}} = \underline{\mathcal{Q}_\Phi(\mathbb{Z}_p)}$ over Sh_{K_Φ}:

$$\mathbb{P}_{K_{\Phi,W^\bullet}} := \underline{\mathrm{Isom}}_{(t_{\alpha,\Phi,\text{ét}}),(s_{\alpha,\Phi})}(W_\bullet\mathbb{L}_{\rho_\Phi,K_\Phi}, W_\bullet V^g_{\mathbb{Z}_p,\mathrm{Sh}_{K_\Phi}}),$$

where the filtration $W_\bullet\mathbb{L}_{\rho_\Phi,K_\Phi}$ comes from the weight filtration $W_\bullet T_p(\mathcal{Q})$ on $T_p(\mathcal{Q})$. Since the weight filtration is constant, both the Hodge tensors and the weight filtration are trivialized when $T_p(\mathcal{Q})$ is trivialized. Therefore, we have a canonical embedding $\mathbb{P}_{K_{\Phi,W^\bullet}} \to \mathbb{P}_{K_{\Phi,G}}$, which induces

$$\underline{\mathcal{G}_\Phi(\mathbb{Z}_p)} \times^{\underline{\mathcal{Q}_\Phi(\mathbb{Z}_p)}} \mathbb{P}_{K_{\Phi,W_\bullet}} = \mathbb{P}_{K_{\Phi,G}}. \tag{6.3}$$

On the other hand, there is a natural pro-étale torsor $\mathbb{P}_{K_\Phi}$ under K_Φ over Sh_{K_Φ}, as defined in Definition 3.13.

Lemma 6.2. *$T_p(\mathcal{Q})$ is trivialized over $\varprojlim_{K'_\Phi \subset K_\Phi} \mathrm{Sh}_{K'_\Phi K^p_\Phi}$.*

Proof. It suffices to show that for each n, $T_p(\mathcal{Q})/p^n$ is trivialized over $\mathrm{Sh}_{K^{(n)}_{\Phi,p}K^p_\Phi}$. It also suffices to work with the Siegel case, where the result follows from the moduli interpretation; see [MP19, §2.2.14]. ☐

Corollary 6.3. *$\mathbb{P}_{K_{\Phi,G}}$ reduces to $\mathbb{P}_{K_\Phi}$.*

Proof. Along the standard representation $\mathcal{P}_\Phi \to \mathrm{GL}(V^g_{\mathbb{Z}_p})$, the push-out torsor of $\mathbb{P}_{K_\Phi}$ is the frame bundle associated with $T_p(\mathcal{Q})$. By Lemma 6.2, we have a morphism

$$\mathbb{P}_{K_\Phi} \to \underline{\mathrm{Isom}}(\mathbb{L}_{\rho_\Phi, K_\Phi}, V^g_{\mathbb{Z}_p, \mathrm{Sh}_{K_\Phi}}),$$

that factors through $\mathbb{P}_{K_\Phi,G}$ and thus factors through $\mathbb{P}_{K_\Phi,W_\bullet}$. ☐

6.1.3. *Hodge-Tate period map.* The de Rham pro-étale torsor $\mathbb{P}_K$ under $\underline{K_p}$ over Sh_K produces a Hodge-Tate period map (3.7), which is the sheaf-theoretic version of the Hodge-Tate period map

$$\mathrm{Sh}_{K^p} \sim \varprojlim_{K_p} \mathrm{Sh}_{K_pK^p} \to \mathcal{F}l_{G,\mu^{-1}}$$

from the perfectoid space Sh_{K^p} to the flag variety $\mathcal{F}l_{G,\mu^{-1}}$. Here $(\mathrm{Sh}_{K^p})^\lozenge = \varprojlim_{K_p} \mathrm{Sh}^\lozenge_{K_pK^p}$. Recall that given a geometric point $x \in \mathrm{Sh}_{K^p}(C, C^+)$, we have an abelian variety A_x over C together with a trivialization $(T_p(A_x), (t_{\alpha,\text{ét},x})) \cong (V_{\mathbb{Z}_p}, (s_\alpha))$, then $\mathrm{HT}_K(x) \in \mathcal{F}l_{G,\mu^{-1}}(C, C^+)$ corresponds to the Hodge-Tate filtration:

$$0 \to (\mathrm{Lie}(A_x))(1) \to T_p(A_x) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}\, A^\vee_x)^\vee \to 0,$$

see [CS17, Thm. 2.1.3].

Similarly, let G_Φ be the $\mathbb{Q}_p$-group $gG_{\mathbb{Q}_p}g^{-1}$. One has a perfectoid space together with a Hodge-Tate period map

$$\mathrm{HT}_{K_\Phi,G} : \mathrm{Sh}_{K^p_\Phi} \to \mathcal{F}l_{G_\Phi,\mu^{-1}_\Phi}, \quad \mathrm{Sh}_{K^p_\Phi} \sim \varprojlim_{K_{\Phi,p}} \mathrm{Sh}_{K_{\Phi,p}K^p_\Phi}, \quad \mathrm{Sh}^\lozenge_{K^p_\Phi} = \varprojlim_{K_{\Phi,p}} \mathrm{Sh}^\lozenge_{K_{\Phi,p}K^p_\Phi},$$

defined as follows: given $x \in \mathrm{Sh}_{K_\Phi}(C, C^+)$, by Corollary 6.3, we have a 1-motive $\mathcal{Q}_x$ over C together with a trivialization $(T_p(\mathcal{Q}_x), (t_{\alpha,\Phi,\text{ét},x})) \cong (V^g_{\mathbb{Z}_p}, (s_{\alpha,\Phi}))$. Then $\mathrm{HT}_{K_\Phi,G}(x) \in \mathcal{F}l_{G_\Phi,\mu^{-1}_\Phi}(C, C^+)$ corresponds to the Hodge-Tate filtration

$$0 \to (\mathrm{Lie}(\mathcal{Q}_x))(1) \to T_p(\mathcal{Q}_x) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}\, \mathcal{Q}^\vee_x)^\vee \to 0.$$

By Corollary 6.3, $\mathrm{HT}_{K_\Phi,G}$ factors through $\mathrm{HT}_{K_\Phi} : \mathrm{Sh}_{K^p_\Phi} \to \mathcal{F}l_{P_\Phi,\mu^{-1}_\Phi}$ and is the sheaf-theoretic version of the Hodge-Tate period map (3.7).

Corollary 6.4. *Over* $\mathrm{Sh}^\lozenge_{K_\Phi}$, *the* $\mathcal{G}_\Phi$*-shtuka associated with* $(\mathbb{P}_{K_\Phi,G}, \mathrm{HT}_{K_\Phi,G})$ *with a leg bounded by* μ_Φ *reduces to the* $\mathcal{P}_\Phi$*-shtuka associated with* $(\mathbb{P}_{K_\Phi}, \mathrm{HT}_{K_\Phi})$ *with a leg bounded by* μ_Φ.

6.1.4. *Comparison.* Let $\mathrm{Sh}_K(G,X)^\lozenge \to \mathrm{Sht}_{\mathcal{G},\mu}$ (resp. $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \to \mathrm{Sht}_{\mathcal{P}_\Phi,\mu_\Phi}$) be the shtuka associated with $\mathbb{P}_K$ (resp. $\mathbb{P}_{K_\Phi}$). We use the notation from subsection 5.2.

Lemma 6.5 (See Corollary 5.13)**.** *We have a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Sh}_K(G,X)^\lozenge \arrow[d] & W^{0,\lozenge} \arrow[l] \arrow[r] & \mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \arrow[d] \\
\mathrm{Sht}_{\mathcal{G},\mu} & & \mathrm{Sht}_{\mathcal{P}_\Phi,\mu_\Phi}. \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"']
\end{tikzcd}$$

Remark 6.6. *This is a special case of Corollary 5.13; see Remark 6.1. We give a more direct and intuitive proof in the Hodge-type case.*

Proof. By Corollary 6.4, it suffices to work with $\mathrm{Sh}_{K_\Phi}(P_\Phi, D_\Phi)^\lozenge \to \mathrm{Sht}_{\mathcal{G}_\Phi,\mu_\Phi}$. In [MP19, Prop. 3.1.6] (using fpqc descent from $\bigsqcup V \to W^0$; see [MP19, §3.1.5] for the notation V), we have comparisons over W^0:

$$\mathcal{Q}[p^\infty] \cong A[p^\infty], \quad T_p(\mathcal{Q}) \cong T_p(A), \quad (t_{\alpha,\text{ét}}) \cong (t_{\alpha,\Phi,\text{ét}}),$$

the tensor comparisons are compatible with $(s_\alpha) \cong (s_{\alpha,\Phi})$, in the sense that the diagram commutes:

$$\begin{tikzcd}
V_{\mathbb{Z}_p}^{\otimes} \arrow[r, "\overset{g^{\otimes}}{\tilde{\to}}"] \arrow[d, "(s_\alpha)"'] & V_{\mathbb{Z}_p}^{g,\otimes} \arrow[r, "\cong"] \arrow[d, "(s_{\alpha,\Phi})"'] & T_p(\mathcal{Q}_{W^0})^{\otimes} \arrow[r, "\cong"] \arrow[d, "(t_{\alpha,\Phi,\text{ét}})"'] & T_p(A_{W^0})^{\otimes} \arrow[d, "(t_{\alpha,\text{ét}})"'] \\
V_{\mathbb{Z}_p}^{\otimes} \arrow[r, "\overset{g^{\otimes}}{\tilde{\to}}"] & V_{\mathbb{Z}_p}^{g,\otimes} \arrow[r, "\cong"] & T_p(\mathcal{Q}_{W^0})^{\otimes} \arrow[r, "\cong"] & T_p(A_{W^0})^{\otimes}.
\end{tikzcd}$$

This gives the isomorphism $\mathrm{Int}(g_\Phi^{-1}) : (\mathbb{P}_K, \mathrm{HT}_K) \cong (\mathbb{P}_{K_\Phi}, \mathrm{HT}_{K_\Phi})$. ☐

6.1.5. *$\mathcal{G}_\Phi$-shtukas.* We move on to integral models.

Proposition 6.7. *The $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G,E}, \phi_{\mathscr{P}_{\Phi,G,E}})$ over $\mathrm{Sh}_{K_\Phi}^{\lozenge} \to \mathrm{Spd}(E)$ with one leg bounded by μ_Φ extends (uniquely) to a $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ over $\mathscr{S}_{K_\Phi}^{\lozenge/} \to \mathrm{Spd}(\mathcal{O}_E)$ with one leg bounded by μ_Φ.*

This proposition follows almost verbatim from [PR24, §4.6.3]. We only need to replace the BKF module of the universal p-divisible group $A[p^\infty]$ over $\mathscr{S}_K$ by the universal p-divisible group $\mathcal{Q}[p^\infty]$ over $\mathscr{S}_{K_\Phi}$. The uniqueness of the extension follows from [PR24, Cor. 2.7.10]. Let us write down some key steps in the construction in order to fix notation.

Proof. By Tannakian formalism, under

$$\rho_\Phi : \mathcal{P}_\Phi \to \mathcal{GSP}(V_{\mathbb{Z}_p}^g) \to \mathrm{GL}(V_{\mathbb{Z}_p}^g),$$

the $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G,E}, \phi_{\mathscr{P}_{\Phi,G,E}})$ over $\mathrm{Sh}_{K_\Phi}^{\lozenge} \to \mathrm{Spd}(E)$ gives a vector shtuka $(\mathscr{V}_{\Phi,E}, \phi_{\mathscr{V}_{\Phi,E}})$ over $\mathrm{Sh}_{K_\Phi}^{\lozenge}$, associated with the de Rham local system $\mathbb{L}_{\rho_\Phi,K_\Phi}$ as in (6.2). We have a family of tensors $(t_{\alpha,\Phi,E}) \in (\mathscr{V}_{\Phi,E}, \phi_{\mathscr{V}_{\Phi,E}})^{\otimes}$, which can be viewed as shtukas homomorphisms over $\mathrm{Sh}_{K_\Phi}^{\lozenge}$:

$$t_{\alpha,\Phi,E} : (\oplus_i \mathscr{V}_{\Phi,E}^{\otimes m_i}, \phi_{\oplus_i \mathscr{V}_{\Phi,E}^{\otimes m_i}}) \to (\oplus_i \mathscr{V}_{\Phi,E}^{\otimes n_i}, \phi_{\oplus_i \mathscr{V}_{\Phi,E}^{\otimes n_i}}). \tag{6.4}$$

Given $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$, a map $S \to \mathscr{S}_{K_\Phi}^{\lozenge}$ is given by an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+}) \to (\widehat{\mathscr{S}_{K_\Phi}})^{\mathrm{ad}}$. We pull back the universal p-divisible group $\mathcal{Q}[p^\infty]$ to $\mathrm{Spec}\, R^{\sharp+}$, and let $M_{\mathrm{inf},\Phi}(R^{\sharp+})$ be the BKF-module of $\mathcal{Q}[p^\infty]$ with one leg at $\phi(\xi) = 0$. Denote $(\mathscr{V}_{\Phi,S}, \phi_{\mathscr{V}_{\Phi,S}})$ by the corresponding minuscule vector shtuka with height $2n$ and dimension n over S with one leg at $S^\sharp$, given by the restriction of $(\phi^{-1})^* M_{\mathrm{inf},\Phi}(R^{\sharp+})$ to $\mathrm{Spa}(W(R^+)) \backslash \{[\varpi] = 0\}$. By [PR24, Thm. 2.7.7], the tensors $(t_{\alpha,\Phi,E})$ in (6.4) extend uniquely to tensors $(t_{\alpha,\Phi}) \in (\mathscr{V}_{\Phi,S}, \phi_{\mathscr{V}_{\Phi,S}})^{\otimes}$.

By exactly the same arguments as in Steps 1, 2, 3 in [PR24, §4.6.3], the v-sheaf over S defined by

$$\overline{\mathscr{P}}_{\Phi,G,S} := \underline{\mathrm{Isom}}_{(t_{\alpha,\Phi}),(s_{\alpha,\Phi})\otimes\mathrm{id}}(\mathscr{V}_{\Phi,S}, V_{\mathbb{Z}_p}^g \otimes_{\mathbb{Z}_p} \mathcal{O}_{S \dot{\times} \mathbb{Z}_p}) \tag{6.5}$$

is induced by a $\mathcal{G}_\Phi$-torsor $\mathscr{P}_{\Phi,G,S}$ over $S \dot{\times} \mathbb{Z}_p$ via $\overline{\mathscr{P}}_{\Phi,G,S} = \overline{\mathcal{G}}_\Phi \times^{\mathcal{G}_\Phi} \mathscr{P}_{\Phi,G,S}$. The Frobenius structure $\phi_{\mathscr{P}_{\Phi,G,S}}$ comes from the Frobenius structure $\phi_{\mathscr{V}_{\Phi,S}}$ of $\mathscr{V}_{\Phi,S}$. Then $(\mathscr{P}_{\Phi,G,S}, \phi_{\Phi,G,S})$ gives a $\mathcal{G}_\Phi$-shtuka over S. It has a leg bounded by μ_Φ by its nature over the generic fiber: one can reduce it to $(C, \mathcal{O}_C)$-points and then apply arguments in [PR24, §3.3.7]. By varying S, and using descent data from the generic fiber, we finish the proof. ☐

6.1.6. *$\mathcal{Q}_\Phi$-shtukas.* We follow the notation from the proof of Proposition 6.7. The BKF module $M_{\mathrm{inf},\Phi}(R^{\sharp+})$ associated with the universal p-divisible group $\mathcal{Q}[p^\infty]$ has a canonical filtration coming from the weight filtration on $\mathcal{Q}[p^\infty]$; therefore, the associated vector shtuka $(\mathscr{V}_{\Phi,S}, \phi_{\mathscr{V}_{\Phi,S}})$ is equipped with a canonical weight filtration $W_\bullet(\mathscr{V}_{\Phi,S}, \phi_{\mathscr{V}_{\Phi,S}})$. Consider the v-sheaf over S defined by

$$\overline{\mathscr{P}}_{\Phi,W_\bullet,S} := \underline{\mathrm{Isom}}_{(t_{\alpha,\Phi}),(s_{\alpha,\Phi})\otimes\mathrm{id}}(W_\bullet \mathscr{V}_{\Phi,S}, W_\bullet V_{\mathbb{Z}_p}^g \otimes_{\mathbb{Z}_p} \mathcal{O}_{S \dot{\times} \mathbb{Z}_p}).$$

Note that $\overline{\mathscr{P}}_{\Phi,W_\bullet,S}$ has a natural $\mathcal{Q}_\Phi$-action factoring through $\overline{\mathcal{Q}}_\Phi$.

Lemma 6.8. *The $\mathcal{G}_\Phi$-torsor $\mathscr{P}_{\Phi,G,S}$ reduces to a $\mathcal{Q}_\Phi$-torsor $\mathscr{P}_{\Phi,W_\bullet,S}$ (i.e. $\mathscr{P}_{\Phi,G,S} = \mathcal{G}_\Phi \times^{\mathcal{Q}_\Phi} \mathscr{P}_{\Phi,W_\bullet,S}$), and $\overline{\mathscr{P}}_{\Phi,W_\bullet,S} = \overline{\mathcal{Q}}_\Phi \times^{\mathcal{Q}_\Phi} \mathscr{P}_{\Phi,W_\bullet,S}$. Moreover, the $\mathcal{G}_\Phi$-torsor isomorphism $\phi_{\mathscr{P}_{\Phi,G,S}}$ reduces to a $\mathcal{Q}_\Phi$-torsor isomorphism*

$$\phi_{\mathscr{P}_{\Phi,W_\bullet,S}} : \mathrm{Frob}_S^*(\mathscr{P}_{\Phi,W_\bullet,S})|_{S\dot{\times}\mathbb{Z}_p\backslash S^\sharp} \xrightarrow{\sim} \mathscr{P}_{\Phi,W_\bullet,S}|_{S\dot{\times}\mathbb{Z}_p\backslash S^\sharp},$$

Proof. This follows from the fact that the weight filtration on $\mathcal{Q}$ is constant. To be more precise, in the construction [PR24, §4.6.3], one first deals with geometric points, and then deals with products of geometric points. Let $T \to S$ be a v-cover such that T is a product of geometric points. By [SW20, Prop. 19.5.3], the category of $\mathcal{G}_\Phi$-torsors (resp. $\mathcal{Q}_\Phi$-torsors) on $S\dot{\times}\mathbb{Z}_p$ is equivalent to the category of $\mathcal{G}_\Phi$-torsors (resp. $\mathcal{Q}_\Phi$-torsors) on $T\dot{\times}\mathbb{Z}_p$ with suitable descent data (resp. descent data for $\mathcal{G}_\Phi$-torsors plus descent data for the weight filtration).

Denote $T = \mathrm{Spa}(B, B^+)$, where $B^+ = \prod_j C_j^+$. The $\mathcal{G}_\Phi$-torsor on $T\dot{\times}\mathbb{Z}_p$ comes from the restriction of the (trivial) $\mathcal{G}_\Phi$-torsor on $A_{\mathrm{inf}}(B^+) = W(B^+)$ given by the (trivial) $\mathcal{G}_\Phi$-BKF-module $\mathcal{T}_{\Phi,G}$ constructed in [PR24, Lem. 4.6.6]. Recall that the weight filtration on $M_{\mathrm{inf},\Phi}$ is the filtration of BKF modules of $0 \subset T[p^\infty] \subset G[p^\infty] \subset \mathcal{Q}[p^\infty]$. Over $W(B^+) = \prod_j W(C_j^+)$, this filtration becomes constant, thus the $\mathcal{G}_\Phi$-torsor $\mathcal{T}_{\Phi,G}$ reduces to a $\mathcal{Q}_\Phi$-torsor $\mathcal{T}_{\Phi,W_\bullet}$. We have descent data of the weight filtration since the weight filtration over T is the one pulled back from S, the $\mathcal{G}_\Phi$-torsor $\mathscr{P}_{\Phi,G,S}$ reduces to a $\mathcal{P}_\Phi$-torsor $\mathscr{P}_{\Phi,W_\bullet,S}$, and $\overline{\mathscr{P}}_{\Phi,W_\bullet,S} = \overline{\mathcal{Q}}_\Phi \times^{\mathcal{Q}_\Phi} \mathscr{P}_{\Phi,W_\bullet,S}$.

The Frobenius structure $\phi_{\mathscr{P}_{\Phi,G,S}}$ comes from the Frobenius structure $\phi_{\mathscr{V}_{\Phi,S}}$ on $\mathscr{V}_{\Phi,S}$, which comes from the restriction of the Frobenius structure $\phi_{M_{\mathrm{inf},\Phi}}$ on the BKF module $M_{\mathrm{inf},\Phi}(R^{\sharp+})$ of $\mathcal{Q}[p^\infty]$. Since the Frobenius structure on $M_{\mathrm{inf},\Phi}(R^{\sharp+})$ preserves the weight filtration, $\phi_{\mathscr{P}_{\Phi,G,S}}$ reduces to the $\mathcal{Q}_\Phi$-torsor isomorphism $\phi_{\mathscr{P}_{\Phi,W_\bullet,S}}$. ☐

Lemma 6.9. *The $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ over $\mathscr{S}_{K_\Phi}^{\lozenge/} \to \mathrm{Spd}(\mathcal{O}_E)$ with one leg bounded by μ_Φ reduces to a $\mathcal{Q}_\Phi$-shtuka $(\mathscr{P}_{\Phi,W^\bullet}, \phi_{\mathscr{P}_{\Phi,W^\bullet}})$ with one leg bounded by μ_Φ.*

Proof. By Lemma 6.8, it suffices to show that the leg of the $\mathcal{Q}_\Phi$-shtuka $(\mathscr{P}_{\Phi,W^\bullet}, \phi_{\mathscr{P}_{\Phi,W^\bullet}})$ is bounded by μ_Φ. It suffices to check the geometric points. Over generic fiber, this follows from Corollary 6.4. Over special fiber, let D be an algebraically closed field, and $x : \mathrm{Spec}\, D \to \mathscr{S}_{K_\Phi}$ be a geometric point on the special fiber, we need to show the pullback $\mathcal{Q}_\Phi$-shtuka $x^*(\mathscr{P}_{\Phi,W_\bullet}, \phi_{\mathscr{P}_{\Phi,W_\bullet}})$ is bounded by μ_Φ. As in [PR24, §4.6.3, Step(3)], we lift $\mathrm{Spa}\, D$ to a point $\tilde{x} : \mathrm{Spa}(C, \mathcal{O}_C) \to (\widehat{\mathscr{S}_{K_\Phi}})^{\mathrm{ad}}$, where C is a complete non-archimedean algebraically closed field of characteristic 0, such that $\mathcal{O}_C/\mathfrak{m}_C = D$. By [PR24, §4.6.3, Step(2)], the $\mathcal{G}_\Phi$-shtuka $\tilde{x}^*(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ on $\mathrm{Spa}(C, \mathcal{O}_C)$ extends to a $\mathcal{G}_\Phi$-BKF module $(\mathcal{T}_{\Phi,G}, \phi_{\mathcal{T}_{\Phi,G}})$ on $W(\mathcal{O}_{C^\flat})$ and then to a $\mathcal{G}_\Phi$-BKF module $(\overline{\mathcal{T}}_{\Phi,G}, \phi_{\overline{\mathcal{T}}_{\Phi,G}})$ on $W(D)$. By the correspondence of $\mathcal{G}_\Phi$-BKF modules on $W(D)$ and $\mathcal{G}_\Phi$-shtukas on $\mathrm{Spec}\, D$ ([PR24, Thm. 2.3.8]), and by [PR24, Prop. 2.4.6], the $\mathcal{G}_\Phi$-shtuka $x^*(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ on $\mathrm{Spec}\, D$ is the specialization of the $\mathcal{G}_\Phi$-shtuka $\tilde{x}^*(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ on $\mathrm{Spa}(C, \mathcal{O}_C)$. In the proof of Lemma 6.8, we see that the $\mathcal{G}_\Phi$-torsor $\mathcal{T}_{\Phi,G}$ on $W(\mathcal{O}_{C^\flat})$ reduces to a $\mathcal{Q}_\Phi$-torsor $\mathcal{T}_{\Phi,W_\bullet}$, thus the $\mathcal{G}_\Phi$-torsor $\overline{\mathcal{T}}_{\Phi,G}$ on $W(D)$ reduces to a $\mathcal{Q}_\Phi$-torsor $\overline{\mathcal{T}}_{\Phi,W_\bullet}$. In particular, the $\mathcal{Q}_\Phi$-shtuka $x^*(\mathscr{P}_{\Phi,W_\bullet}, \phi_{\mathscr{P}_{\Phi,W_\bullet}})$ on $\mathrm{Spec}\, D$ is the specialization of the $\mathcal{Q}_\Phi$-shtuka $\tilde{x}^*(\mathscr{P}_{\Phi,W_\bullet}, \phi_{\mathscr{P}_{\Phi,W_\bullet}})$ on $\mathrm{Spa}(C, \mathcal{O}_C)$.

In particular, the relative position of $\phi_{\mathscr{P}_{\Phi,W^\bullet}}(\mathrm{Frob}^*(\mathscr{P}_{\Phi,W^\bullet}))$ and $\mathscr{P}_{\Phi,W^\bullet}$ at $\tilde{x}$ gives a $\mathrm{Spa}(C, \mathcal{O}_C)$-point in $\mathrm{Gr}_{Q_\Phi,\mu_\Phi^{-1}}$, which lifts to a $\mathrm{Spa}(\mathcal{O}_C)$-point in $\mathrm{Gr}_{\mathcal{Q}_\Phi}$ whose specialization gives the relative position of $\phi_{\mathscr{P}_{\Phi,W^\bullet}}(\mathrm{Frob}^*(\mathscr{P}_{\Phi,W^\bullet}))$ and $\mathscr{P}_{\Phi,W^\bullet}$ at x.

By construction, $\mathbb{M}_{\mathcal{Q}_\Phi,\mu_\Phi} = \underline{|\,\mathrm{Gr}_{Q_\Phi,\mu_\Phi^{-1}}|^{wgc}} \times_{\underline{|\,\mathrm{Gr}_{\mathcal{Q}_\Phi,\mathcal{O}_E}|}} \mathrm{Gr}_{\mathcal{Q}_\Phi,\mathcal{O}_E}$ (where $|\,\mathrm{Gr}_{Q_\Phi,\mu_\Phi^{-1}}|^{wgc}$ is the weakly generalizing closure of $|\,\mathrm{Gr}_{Q_\Phi,\mu_\Phi^{-1}}|$ in $|\,\mathrm{Gr}_{\mathcal{Q}_\Phi,\mathcal{O}_E}|$). Since $|\,\mathrm{Spa}(C, \mathcal{O}_C)|$ is open and dense in $|\,\mathrm{Spa}(\mathcal{O}_C)|$ and the image of $|\,\mathrm{Spa}(\mathcal{O}_C)|$ in $|\,\mathrm{Gr}_{\mathcal{Q}_\Phi,\mathcal{O}_E}|$ is weakly generalizing, then $\mathrm{Spa}(\mathcal{O}_C) \to \mathrm{Gr}_{\mathcal{Q}_\Phi}$ factors through $\mathbb{M}_{\mathcal{Q}_\Phi,\mu_\Phi}$, the $\mathcal{Q}_\Phi$-shtuka $x^*(\mathscr{P}_{\Phi,W_\bullet}, \phi_{\mathscr{P}_{\Phi,W_\bullet}})$ on $\mathrm{Spec}\, D$ is bounded by μ_Φ. ☐

6.1.7. *$\mathscr{P}_\Phi$-shtukas.*

Proposition 6.10. *The $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ over $\mathscr{S}_{K_\Phi}^{\diamondsuit/} \to \mathrm{Spd}(\mathcal{O}_E)$ with one leg bounded by μ_Φ reduces to a $\mathcal{P}_\Phi$-shtuka $(\mathscr{P}_\Phi, \phi_{\mathscr{P}_\Phi})$ with one leg bounded by μ_Φ. This extends Corollary 6.4 over the integral base.*

Proof. By Lemma 6.8 and 6.9, the $\mathcal{G}_\Phi$-shtuka $(\mathscr{P}_{\Phi,G}, \phi_{\mathscr{P}_{\Phi,G}})$ with one leg bounded by μ_Φ reduces to the $\mathcal{Q}_\Phi$-shtuka $(\mathscr{P}_{\Phi,W^\bullet}, \phi_{\mathscr{P}_{\Phi,W^\bullet}})$ with one leg bounded by μ_Φ. Corollary 6.4 says that the $\mathcal{Q}_\Phi$-shtuka $(\mathscr{P}_{\Phi,W^\bullet,E}, \phi_{\mathscr{P}_{\Phi,W^\bullet,E}})$ with one leg bounded by μ_Φ reduces to a $\mathcal{P}_\Phi$-shtuka $(\mathscr{P}_{\Phi,E}, \phi_{\mathscr{P}_{\Phi,E}})$ with one leg bounded by μ_Φ. Then the result follows from Proposition 1.43. ☐

6.1.8. *Rest of axioms.*

Proposition 6.11. *Let (G, X) be a Hodge-type Shimura datum, and let $\mathcal{G}$ be any quasi-parahoric model of $G_{\mathbb{Q}_p}$. Then there exists a family of integral models $\{\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)\}_{K_\Phi^p}$ that is canonical in the sense of Definition 4.2, and is adapted with $P_\Phi \to G_{\Phi,h}$ in the sense of Definition 4.6.*

As in Remark 6.1, $P_\Phi = P_\Phi^c$, so it is trivially true that the integral model is adapted with $P_\Phi \to P_\Phi^c$ in the sense of 4.5.

Proof. It suffices to deal with stabilizer quasi-parahoric $\mathcal{G}$ by the functoriality result (Proposition 4.9, case (2)). Let $\mathcal{G}$ be a stabilizer quasi-parahoric. We show that the family of integral models $\{\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)\}_{K_\Phi^p}$ defined via relative normalizations is canonical. This follows from the Siegel case (Lemma 6.12) and the functoriality result (Proposition 4.9, case (1)). Note that the condition (b) is not automatic, we verified it in [Mao25a, Prop. 1.2(2)] under a fixed adjusted Hodge embedding. Also note that, for the integral canonical model $\mathscr{S}_K(G, X)$ of a Siegel-type Shimura variety $\mathrm{Sh}_K(G, X)$ with K_p being hyperspecial, $\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ is again the integral canonical model of the Siegel-type Shimura variety $\mathrm{Sh}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ with $K_{\Phi,h}$ being hyperspecial, for each $\Phi \in \mathcal{CLR}(G, X)$, see [Mao25a, Thm. 3.58(2)]. ☐

Lemma 6.12. *Proposition 6.11 holds when $(G, X) = (G^\ddagger, X^\ddagger)$ is a Siegel Shimura datum and when $\mathcal{G}$ is hyperspecial.*

Proof. In this case, we have moduli interpretations for the smooth integral models $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)$ and $\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ ([MP19, §2.2.14, 2.2.15]), and a projection $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi) \to \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ defined via the forgetful functor ([MP19, §2.2.16]); then $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)$ is adapted with $P_\Phi \to G_{\Phi,h}$. Now we verify those axioms in Definition 4.2.

Axiom (1) follows from the Néron-Ogg-Shafarevich criterion for 1-motives; see [MP19, Appendix, Lem. A.3.5]. Indeed, recall that $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)$ represents the following moduli functor: given a $\mathbb{Z}_p$-scheme S, $\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)(S)$ is the set of isomorphism classes of tuples $(\mathcal{Q}, \lambda, \alpha, \alpha^\vee, \epsilon)$ over S, where $(\mathcal{Q}, \lambda)$ is a polarized 1-motive over S, equipped with isomorphisms of sheaves of $\mathbb{Z}$-modules over S:

$$\alpha : \mathrm{Gr}_0^W \underline{V}^g(\mathbb{Z}) \cong \mathcal{Q}^{\text{ét}}, \quad \alpha^\vee : \mathrm{Gr}_0^W \underline{V}^{\vee,g}(\mathbb{Z})(\nu) \cong \mathcal{Q}^{\vee,\text{ét}} = \mathcal{Q}^{\text{mult}},$$

where $V^\vee(\nu)$ is the dual representation $V^\vee$ twisted by the similitude character $\nu : \mathrm{GSp}(V, \psi) \to \mathbb{G}_{\mathrm{m}}$, and $\epsilon \in H^0(S, \underline{\mathrm{Isom}}(V^g(\hat{\mathbb{Z}}^p), \hat{T}^p(\mathcal{Q}))/K_{\Phi,G}^p)$ is a $K_{\Phi,G}^p$-level structure, where $K_{\Phi,G}^p = g^p K^p g^{p,-1}$. See [MP19, §2.2.14] for conventions.

Let $x \in \mathscr{S}_{K_{\Phi,p}}(P_\Phi, D_\Phi)(R[1/p])$, it gives a tuple $(\mathcal{Q}_\eta, \lambda_\eta, \alpha_\eta, \alpha_\eta^{-1}, \epsilon_\eta)$ over $\mathrm{Spec}\, R[1/p]$. By [MP19, Appendix, Lem. A.3.5], $\mathcal{Q}_\eta$ (resp. $\mathcal{Q}_\eta^\vee$) extends to a 1-motif $\mathcal{Q}$ (resp. $\mathcal{Q}^\vee$) over $\mathrm{Spec}\, R$. Note that α_η, α_η^{-1}, ϵ_η extend to α, α^{-1}, ϵ respectively by the étaleness, and λ_η extends to λ using the rigidity theorem for homomorphisms of semi-abelian schemes (thus of 1-motives), see [FC90, I, Prop. 2.7]. The tuple $(\mathcal{Q}, \lambda, \alpha, \alpha^{-1}, \epsilon)$ over $\mathrm{Spec}\, R$ gives the wanted lifting.

Axiom (2): This follows from moduli interpretations. Also see [Wu25, Lem. 4.25].

Axiom (3): This is a special case of Proposition 6.10.

Axiom (4): We replace the Serre-Tate theorem for abelian schemes with the Serre-Tate theorem for 1-motives. Fix a k-point $x \in \mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)(k)$. We have a tuple $(\mathcal{Q}_0, \lambda_0, \alpha_0, \alpha_0^\vee, \epsilon_0)$ over $\operatorname{Spec} k$, the formal completion $\hat{S}_x := (\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi))^\wedge_{/x}$ represents the functor $\operatorname{Def} : \operatorname{Art}_{W(k)} \to \operatorname{Set}$ sending S to isomorphism classes of deformations over S of the tuple $(\mathcal{Q}_0, \lambda_0, \alpha_0, \alpha_0^\vee, \epsilon_0)$.

On the other hand, $\mathcal{M}^{\mathrm{int}} := \mathcal{M}^{\mathrm{int}}_{\mathcal{G}_\Phi, b_{\Phi,x}, \mu_\Phi}$ is a Rapoport-Zink space, the deformation space of p-divisible groups, for example, see [SW20, Thm. 25.1.2]. Let (X_1, λ_1) be a polarized p-divisible group in the isogeny class determined by $[b_{\Phi,x}] \in B(G, \{\mu^{-1}\}) = B(G_\Phi, \{\mu_\Phi^{-1}\})$, given a $\mathbb{Z}_p$-scheme S, $\mathcal{M}^{\mathrm{int}}(S)$ is the set of isomorphism classes of (X, λ, ρ) over S, where (X, λ) is a polarized p-divisible group over S and $\rho : (X, \lambda) \times_S S/p \to (X_1, \lambda_1) \times_k S/p$ is a quasi-isogeny. Let $x_0 \in \mathcal{M}^{\mathrm{int}}(k)$ be the base-point associated with $(X_1, \lambda_1, \mathrm{id})$. Since the quasi-isogeny ρ over S/p is rigid in the sense that there is a unique ρ lifting the isomorphism $\rho \otimes S/\mathfrak{m}_S$, for example, see [Kim18, §2.3]. Therefore, $(\mathcal{M}^{\mathrm{int}})^\wedge_{/x_0}$ represents the functor $\operatorname{Def}_{(X_1, \lambda_1)} : \operatorname{Art}_{W(k)} \to \operatorname{Set}$ sending S to isomorphism classes of deformations over S of the polarized p-divisible group (X_1, λ_1).

Let (X_0, λ_0) be the p-divisible group $(\mathcal{Q}_0, \lambda_0)[p^\infty]$, then we have an isogeny $\rho_{01} : (X_0, \lambda_0) \to (X_1, \lambda_1)$. (X_0, λ_0, ρ) gives a k-point in $\mathcal{M}^{\mathrm{int}}$, which we denote by x_1.

By the main theorem of [BM19], the Serre-Tate theorem holds for 1-motives: let $R \in \operatorname{Art}_{W(k)}$, $\mathcal{M}_1(R)$ be the category of 1-motives over R. Let M (resp. B) be a 1-motive (resp. a p-divisible group) over R, M_0 (resp. B_0) be its reduction over $\operatorname{Spec} k$. Let $\operatorname{Def}(R, k)$ be the category of triples (M_0, B, e), where M_0 is a 1-motive over $\operatorname{Spec} k$, B is a p-divisible group over R, and $e : B_0 \to M_0[p^\infty]$ is an isomorphism of p-divisible groups. There is a natural equivalence of categories:

$$\Delta_R : \mathcal{M}_1(R) \to \operatorname{Def}(R, k), \quad M \to (M_0, M[p^\infty], \text{natural } e).$$

Consider the morphism $\hat{S}_x \to (\mathcal{M}^{\mathrm{int}})^\wedge_{/x_1}$ defined by sending $(\mathcal{Q}, \lambda, \alpha, \alpha^\vee, \epsilon)$ to $((\mathcal{Q}, \lambda)[p^\infty], \mathrm{id})$. We claim this is an equivalence of categories. By Serre-Tate for 1-motives, and by the fact that $(\alpha, \alpha^\vee)$ and ϵ are rigid by the étaleness (thus determined by their reductions $(\alpha_0, \alpha_0^\vee)$ and ϵ_0 over $\operatorname{Spec} k$ respectively), it suffices to show that, given a polarization of p-divisible groups $\lambda[p^\infty] : \mathcal{Q}[p^\infty] \to \mathcal{Q}^\vee[p^\infty]$, we can lift it canonically to a polarization of 1-motives $\lambda : \mathcal{Q} \to \mathcal{Q}^\vee$. Recall that a polarization $\lambda : \mathcal{Q} \to \mathcal{Q}^\vee$ is a morphism between 1-motives such that $\lambda^{\mathrm{ab}} : \mathcal{Q}^{\mathrm{ab}} \to \mathcal{Q}^{\vee,\mathrm{ab}}$ is a polarization of abelian schemes, $\lambda^{\text{ét}} : \mathcal{Q}^{\text{ét}} \to \mathcal{Q}^{\mathrm{mult}}$ is injective, and $(\lambda^{\text{ét}})^\vee = \lambda^{\mathrm{mult}}$. The full-faithfulness of Δ_R gives a unique morphism $\lambda : \mathcal{Q} \to \mathcal{Q}^\vee$ and its proof also shows that λ^{ab} is a polarization, $\lambda^{\text{ét}}$ and λ^{mult} satisfy the properties since they are étale in nature and $\lambda_0^{\text{ét}}$ and $\lambda_0^{\mathrm{mult}}$ do satisfy the properties.

Therefore, we have an isomorphism $\hat{S}_x \cong (\mathcal{M}^{\mathrm{int}})^\wedge_{/x_1}$. We show that the isomorphism $\hat{S}_x \cong (\mathcal{M}^{\mathrm{int}})^\wedge_{/x_1}$ matches the supported shtukas. By [SW20, Thm. 25.1.2], the identification of the Rapoport-Zink space with $\mathcal{M}^{\mathrm{int}}$ is to evaluate the universal p-divisible group at perfectoid space $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}$ to get a BKF module over $A_{\mathrm{inf}}(R, R^+)$, and then restrict the BKF module to $S \dot{\times} \mathbb{Z}_p$ to get the desired shtuka; also, the quasi-isogeny ρ provides a framing. Under this identification, since $\hat{S}_x \cong (\mathcal{M}^{\mathrm{int}})^\wedge_{/x_1}$ matches the universal p-divisible groups, it matches the supported shtukas. Finally, after changing the base point, $(\mathcal{M}^{\mathrm{int}})^\wedge_{x_1} \cong (\mathcal{M}^{\mathrm{int}})^\wedge_{x_0}$, we are done. □

Finally, to prove the axiom (1) in abelian-type case, we need following enhanced result:

Lemma 6.13. *Keep the notation from Proposition 6.11. Let l be a prime $\neq p$, $K_\Phi^{l,p} = K_\Phi^p \cap P_\Phi(\mathbb{A}_f^{p,l})$, $K_\Phi^l = K_{\Phi,p} K_\Phi^{p,l}$. For every discrete valuation ring R of mixed characteristic over $\mathcal{O}_E$, we have*

$$\operatorname{Sh}_{K_\Phi^l}(P_\Phi, D_\Phi)(R[1/p]) = \mathscr{S}_{K_\Phi^l}(P_\Phi, D_\Phi)(R).$$

Proof. This follows from the proof of the Siegel case (Lemma 6.12) and the proof of the functoriality result (Proposition 4.9, case (1)). Note that in [MP19, Appendix, Lem. A.3.5], one only needs to trivialize $T^l(\mathcal{Q}_\eta)$, rather than the whole $\hat{T}^p(\mathcal{Q}_\eta)$. □

6.2. **Abelian-type case.**

6.2.1. *Construction.* Let us recall the construction in [Wu25]. The key steps are listed in Construction 6.14, 6.15, and 6.16 for convenience.

Construction 6.14. Let (G_2, X_2) be an abelian-type Shimura datum with an associated Hodge-type Shimura datum (G, X); note that, in [Wu25], it was denoted by (G_0, X_0). There is a central isogeny $\pi^{\mathrm{der}} : G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ with kernel C^{der}.

Let $G' := G_2 \times^{G_2^{\mathrm{der}}} (G/C^{\mathrm{der}})$. There is an embedding $\pi^a : G_2 \to G'$ and a map $\pi^b : G \to G'$ such that they induce morphisms between Shimura data $\pi^a : (G_2, X_2) \to (G', X_a)$ and $\pi^b : (G, X) \to (G', X_b)$. By replacing (G, X) with a conjugate by $g \in G^{\mathrm{ad}}(\mathbb{Q})$, we assume that the images of X_2 and X in X^{ad} have nontrivial intersection. We then have that the images of X_a and X_b in X^{ad} coincide. Then π^a and π^b satisfy the conditions in [Wu25, §1.4.3]. See [Wu25, §1.4.3 and Const. 1.46].

We recall more group-theoretic preparations.

Construction 6.15. We start with $K_2 = K_{2,p}K_2^p \subset G_2(\mathbb{A}_f)$ and an admissible cone decomposition Σ_2 for (G_2, X_2, K_2).

- By [Wu25, Prop. 1.35, Prop. 1.37 and Prop. 1.47], we can find $K' := K'_p K'^{,p} \subset G'(\mathbb{A}_f)$ containing K_2, and an admissible smooth projective ZP-invariant cone decomposition Σ for (G', X_a, K') and (G', X_b, K') inducing a cone decomposition Σ'_2 refining Σ_2 for (G_2, X_2, K_2), such that the strata between toroidal (with cones chosen as above) and minimal compactifications of (G_2, X_2) and (G', X_a) are all open and closed embeddings, and such that the map between mixed Shimura varieties for any cusp label representative $\Phi_2 \in \mathcal{CLR}(G_2, X_2)$ mapping to a cusp label representative $\Phi \in \mathcal{CLR}(G', X_a)$ is an isomorphism. (Note that K'_p is chosen such that $K'_p \cap G_2(\mathbb{Q}_p) = K_{2,p}$.) There is an identification $\mathcal{CLR}(G', X_a) = \mathcal{CLR}(G', X_b)$ by definition and construction.
- Denote $I_{G'/G,K'} := \mathrm{Stab}_{G'(\mathbb{Q})}(X)\pi^b(G(\mathbb{A}_f))\backslash G'(\mathbb{A}_f)/K'$. Choose a complete system $\{g_\alpha\}_{\alpha \in I_{G'/G,K'}}$ of representatives of $I_{G'/G,K'}$ in $G'(\mathbb{A}_f)$. Let $K_p^\alpha := \pi^{b,-1}(g_\alpha K'_p g_\alpha^{-1}) \subset G(\mathbb{Q}_p)$. Choose suitable neat open compact $K^{\alpha,p} \subset G(\mathbb{A}_f^p)$, and denote $K^\alpha := K_p^\alpha K^{\alpha,p}$. We obtain the induced cone decompositions $\Sigma^\alpha := \pi^{b,*}(\Sigma)$.
- In our case, we assume that K'_p is a quasi-parahoric subgroup, which is the $\mathbb{Z}_p$-points of a quasi-parahoric group scheme $\mathcal{G}'$ corresponding to $x \in \mathcal{B}(G', \mathbb{Q}_p)$.

  Following the construction in Case (STB) of [Wu25, §4.2], there is a Siegel-type Shimura datum $(G^\ddagger, X^\ddagger) = (\mathrm{GSp}(V, \psi), X^\ddagger)$ with an embedding $\rho : (G, X) \hookrightarrow (G^\ddagger, X^\ddagger)$ obtained as follows:

  For any α, we choose $K_p^{\ddagger,\alpha} \subset G^\ddagger(\mathbb{Q}_p)$ and neat open compact $K^{\ddagger,\alpha,p} \subset G^\ddagger(\mathbb{A}_f^p)$ such that

  $$\rho^\alpha : (G, X, K_p^\alpha) \hookrightarrow (G^{\ddagger,\alpha}, X^{\ddagger,\alpha}, K_p^{\ddagger,\alpha})$$

  is an adjusted Hodge embedding satisfying the setup in both [MP19, §3.1] and [DvHKZ26].

  In fact, $G^{\ddagger,\alpha} := \mathrm{GSp}(W^\alpha, \psi^\alpha)$, $K_p^{\ddagger,\alpha,\sharp} = \mathrm{Stab}_{G^{\ddagger,\alpha}(\mathbb{Q}_p)}(W^\alpha_{\mathbb{Z}_p})$ for some self-dual $\mathbb{Z}_p$-lattice $W^\alpha_{\mathbb{Z}_p}$ in $W^\alpha_{\mathbb{Q}_p}$. Moreover, $W^\alpha_{\mathbb{Z}_p}$ comes from a self-dual $\mathbb{Z}$-lattice $W^\alpha_{\mathbb{Z}}$.

  Furthermore, there are positive integers n_α such that $(W^\alpha, \psi^\alpha)^{\perp n_\alpha} = (V, \psi)$ for a symplectic space (V, ψ) for all α. By defining $V^\alpha_? := W^{\alpha,\perp n_\alpha}_?$ for $? = \mathbb{Q}, \mathbb{Z}, \mathbb{Z}_p$, $(G^\ddagger, X^\ddagger)$ as the Siegel Shimura datum defined by (V, ψ), and $K_p^{\ddagger,\alpha} := \mathrm{Stab}_{G^\ddagger(\mathbb{Q}_p)}(V^\alpha_{\mathbb{Z}_p})$, we obtain from ρ^α the Hodge embeddings

  $$\iota^\alpha : (G, X, K_p^\alpha) \hookrightarrow (G^\ddagger, X^\ddagger, K_p^{\ddagger,\alpha}).$$

  After this slight modification, [Mao25a, Thm. 3.58] still holds.

- By [Wu25, Prop. 1.47] again, cone decompositions $\Sigma^{\ddagger,\alpha}$ can be chosen for $(G^{\ddagger}, X^{\ddagger}, K^{\ddagger,\alpha})$ such that, for any α, Σ^{α} is induced by both Σ via $\pi^b(g_\alpha)$ and $\Sigma^{\ddagger,\alpha}$ via ι^α, and such that $\mathscr{S}^{\Sigma^\alpha}_{K^\alpha}$ can be constructed satisfying Axiom 5.1 according to [MP19].

We now recall some schematic constructions in [Wu25].

Construction 6.16. We recall the construction of integral models of boundary mixed Shimura varieties $\mathscr{S}_{K_{\Phi_2}}$ for $\Phi_2 \in \mathcal{CLR}(G_2, X_2)$ following [Wu25, Sec. 4.2] in Case (STB) and its normalization (DL) case to parahoric levels.

(1) We introduce more group-theoretic setup related to Bruhat-Tits theory. In general, there are maps between groups $\pi^a : K_{2,p} \hookrightarrow K'_p$ and $\pi^b(g_\alpha) : K^\alpha_p \to g_\alpha K'_p g_\alpha^{-1}$. Starting with $\Phi_2 \in \mathcal{CLR}(G_2, X_2)$, let Φ be the image of Φ_2 in $\mathcal{CLR}(G', X_a) \cong \mathcal{CLR}(G', X_b)$.

Fix a point in the reduced building $x \in \mathcal{B}(G_2, \mathbb{Q}_p)$. We denote by $K^\circ_{2,p}$ (resp. $K^{\mathrm{stb}}_{2,p}$ and $K_{2,p}$) the parahoric (resp. the stabilizer quasi-parahoric and resp. a general quasi-parahoric) subgroup of $G_2(\mathbb{Q}_p)$ corresponding to x. For Φ_2, define $K^{\min}_{\Phi_2,p} := P_{\Phi_2}(\mathbb{Q}_p) \cap g_{\Phi_2} K^\circ_{2,p} g_{\Phi_2}^{-1}$ (resp. $K^{\mathrm{stb}}_{\Phi_2,p} := P_{\Phi_2}(\mathbb{Q}_p) \cap g_{\Phi_2} K^{\mathrm{stb}}_{2,p} g_{\Phi_2}^{-1}$ and resp. $K_{\Phi_2,p} := P_{\Phi_2}(\mathbb{Q}_p) \cap g_{\Phi_2} K_{2,p} g_{\Phi_2}^{-1}$); note that $K^{\mathrm{stb}}_{\Phi_2,p}$ is still a stabilizer quasi-parahoric subgroup. Let $K^{\prime,\mathrm{stb}}_p$ be the stabilizer quasi-parahoric subgroup in $G'(\mathbb{Q}_p)$ corresponding to x. Similarly, define $K^{\mathrm{stb}}_{\Phi,p} := P_\Phi(\mathbb{Q}_p) \cap g_{\Phi_2} K^{\prime,\mathrm{stb}}_p g_{\Phi_2}^{-1}$ and $\widetilde{K}^{\mathrm{stb}}_{\Phi,p} := ZP_\Phi(\mathbb{Q}_p) \cap g_{\Phi_2} K^{\prime,\mathrm{stb}}_p g_{\Phi_2}^{-1}$. Note that we can still use Definition 1.13 and Lemma 3.3 to get that $\widetilde{K}^{\mathrm{stb}}_{\Phi,p}$ is a stabilizer quasi-parahoric subgroup. The prime-to-p neat open compact subgroups are chosen and written in an obvious pattern.

(2) Denote by (P_Φ, D^b_Φ) (resp. (P_Φ, D^a_Φ)) the mixed Shimura datum associated with $\Phi \in \mathcal{CLR}(G, X_b)$. We construct $\{\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)\}_{K^{\prime,p}}$ and $\{\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)\}_{K^{\prime,p}}$. Note that the Shimura datum (G, X) is chosen such that any place v' of $\mathbb{E}' := \mathbb{E}(G, X) \cdot \mathbb{E}(G_2, X_2)$ over a place v_2 of $\mathbb{E}_2 := \mathbb{E}(G_2, X_2)$ over p splits completely over v_2.

We define $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$ over $\mathcal{O}' := \mathcal{O}_{\mathbb{E}'} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E},(v)}$ for any place v of $\mathbb{E} := \mathbb{E}(G, X)$. For any cusp label $\Phi \in \mathcal{CLR}(G', X_b)$ extending to a ZP-cusp label $[ZP^b(\Phi)]$, by [Wu25, (4.22)], we construct

$$\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi) := \coprod_{\alpha \in I_{G'/G,K^{\prime,\mathrm{stb}}}} \coprod_{\pi^b(g^\alpha_0)\alpha \sim g^b} \coprod_{[\sigma^\alpha] \in [\sigma]_{ZP}} \mathscr{S}_{K_{\Phi^\alpha_0}} / \Delta_{\Phi^\alpha_0, K^{\prime,\mathrm{stb}}}(ZP_\Phi). \tag{6.6}$$

The group $\Delta_{\Phi^\alpha_0, K^{\prime,\mathrm{stb}}}(ZP_\Phi)$ is a group that acts on $\mathscr{S}_{K_{\Phi^\alpha_0}}$ through a finite group; this group is defined and functorial in *all* neat $K^{\prime,\mathrm{stb}}$; its quotient is finite étale by [Wu25, Lem. 4.25]. Writing in this form a priori depends on the choice of $\mathscr{S}^{\Sigma^\alpha}_{K^\alpha}$ and, in particular, on the choice of Σ^α. But we can also re-label the disjoint union in (6.6) as

$$\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi) := \coprod_{\beta \in I_{\Phi,K^{\prime,\mathrm{stb}}}} \mathscr{S}_{K_{\Phi_\beta}} / \Delta_{\Phi_\beta, K^{\prime,\mathrm{stb}}}(ZP_\Phi), \tag{6.7}$$

where Φ_β are cusp label representatives in $\mathcal{CLR}(G, X)$ that occurred in (6.6) and the index set $I_{\Phi,K^{\prime,\mathrm{stb}}}$ is defined in [Wu25, 4.2.6]; it depends only on Φ, $K^{\prime,\mathrm{stb}}$, the Shimura data (G, X) and the group G', satisfying obvious functoriality for varying $K' \subset G'(\mathbb{A}_f)$. The level group K_{Φ_β} is defined as $P_{\Phi_\beta}(\mathbb{A}_f) \cap g_{\Phi_\beta} K^\alpha g_{\Phi_\beta}^{-1}$, where $K^\alpha_p = \pi^{b,-1}(g_\alpha K^{\prime,\mathrm{stb}}_p g_\alpha^{-1})$.

Replacing $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$ with $\mathscr{S}_{\overline{\widetilde{K}}{}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$ (resp. $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi,h}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$), the integral model is constructed by replacing the Hodge-type mixed Shimura varieties $\mathscr{S}_{K_{\Phi_\beta}}$ in (6.7) with $\mathscr{S}_{\overline{K}_{\Phi_\beta}}$ (resp. $\mathscr{S}_{K_{\Phi_{\beta},h}}$). Again, the group action of Δ_{Φ_β} factors through a finite group that acts freely on $\mathscr{S}_{\overline{K}_{\Phi_\beta}}$ (resp. $\mathscr{S}_{K_{\Phi_\beta,h}}$). Moreover, there is a sequence of morphisms

$$\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi) \to \mathscr{S}_{\overline{\widetilde{K}_{\Phi}}^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi) \to \mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi,h}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi). \tag{6.8}$$

The first morphism is a torsor under a split torus $\mathbf{E}_{\widetilde{K}_\Phi}$, and the second morphism is an abelian scheme torsor by [Wu25, Prop. 4.30].

To construct $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)$, we first base change $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$ to $\mathscr{O}^{ur}$ such that $\mathscr{O}^{ur} = \mathscr{O}_{\mathbb{E}''} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathscr{O}_{\mathbb{E},(v)}$ where $\mathbb{E}''$ is finite extension of $\mathbb{E}'$ that is unramified over p, and that the generic fiber of the base change $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)_{\mathscr{O}^{ur}}$ is $\mathrm{Sh}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)_{\mathbb{E}''}$. Thus, the integral model $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)$ is obtained from desending the model $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)_{\mathscr{O}^{ur}}$ with the descend datum given by the Shimura datum (G, X_a). There is also a sequence of morphisms as (6.8) replacing "b" with "a" having the same property.

(3) Let K'_p be a quasi-parahoric subgroup associated with $K^{\prime,\mathrm{stb}}_p$. Denote $\widetilde{K}'_{\Phi,p} = ZP_\Phi(\mathbb{A}_f) \cap g_\Phi K'_p g_\Phi^{-1}$. We construct $\{\mathscr{S}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)\}_{K^{\prime,p}}$ as the normalization from the tower $\{\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)\}_{K^{\prime,p}}$. More explicitly, we also construct $\mathscr{S}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$ as $\coprod_{\beta' \in I_{\Phi,K'}} \mathscr{S}_{K_{\Phi_{\beta'}}}/\Delta_{\Phi_{\beta'},K'}(ZP_\Phi)$. Each $\mathscr{S}_{K_{\Phi_{\beta'}}}$ is constructed as follows: For Any $\Phi_{\beta'} \in \mathcal{CLR}(G, X)$, the integral model $\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_{\beta'}}}$ at stabilizer quasi-parahoric level is constructed as before. Then $\mathscr{S}_{K_{\Phi_{\beta'}}}$, the integral model assciated with the same cusp label representative but at the quasi-parahoric level, is defined by taking relative normalization from $\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi_{\beta'}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$. There is a finite map

$$\mathscr{S}_{K_{\Phi_{\beta'}}}/\Delta_{\Phi_{\beta'},K'}(ZP_\Phi) \to \mathscr{S}_{K^{\mathrm{stb}}_{\Phi_{\beta'}}}/\Delta_{\Phi_{\beta'},K^{\prime,\mathrm{stb}}}(ZP_\Phi).$$

There is also a sequence as (6.8) for $\mathscr{S}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^b_\Phi)$.

We can also construct $\mathscr{S}_{\widetilde{K}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)$ with the same method as above. See [Wu25, Prop. 4.30 and Const. 4.33].

(4) We construct $\{\mathscr{S}_{K^{\min}_{\Phi_2}}\}_{K^p_2}$, $\{\mathscr{S}_{K_{\Phi_2}}\}_{K^p_2}$ and $\{\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}}\}_{K^p_2}$. In fact, they are constructed by taking the relative normalization of $\{\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_{\Phi}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)\}_{K^{\prime,p}}$ in $\{\mathrm{Sh}_{K^{\min}_{\Phi_2}}\}_{K^p_2}$, $\{\mathrm{Sh}_{K_{\Phi_2}}\}_{K^p_2}$ and $\{\mathrm{Sh}_{K^{\mathrm{stb}}_{\Phi_2}}\}_{K^p_2}$, respectively.

□

6.2.2. *Choosing accessible Hodge-type liftings.*

Definition 6.17. *Fix a prime number p. Let (G, X) and (G_1, X_1) be Shimura data with a central isogeny $\pi : G^{\mathrm{der}} \to G_1^{\mathrm{der}}$ such that π induces an isomorphism between the two adjoint Shimura data. Let $\mathcal{G}^\circ_{1,x}$ be a parahoric group scheme corresponding to x in the building $\mathcal{B}(G_1, \mathbb{Q}_p)$, and $\mathcal{G}^\circ_x$ be the corresponding parahoric group scheme of $G_{\mathbb{Q}_p}$.*

*We say that $\breve{K}^\circ_p := \mathcal{G}^\circ_x(\breve{\mathbb{Z}}_p)$ **is accessible to** $\breve{K}^\circ_{1,p} := \mathcal{G}^\circ_{1,x}(\breve{\mathbb{Z}}_p)$ if the map $\pi : G^{\mathrm{der}}(\breve{\mathbb{Q}}_p) \to G_1^{\mathrm{der}}(\breve{\mathbb{Q}}_p)$ restricts to a map*

$$\breve{K}^\circ_p \cap G^{\mathrm{der}}(\breve{\mathbb{Q}}_p) \to \breve{K}^\circ_{1,p} \cap G_1^{\mathrm{der}}(\breve{\mathbb{Q}}_p). \tag{6.9}$$

*We say that (G, X) **is accessible to** (G_1, X_1) (at p with respect to π) if, for any point x in the reduced building of $G_{1,\mathbb{Q}_p}$, $\breve{K}^\circ_p$ is always accessible to $\breve{K}^\circ_{1,p}$.*

Convention 6.18. *Let H and G be two connected reductive groups over $\mathbb{Q}$ or $\breve{\mathbb{Q}}_p$ with a map $f : H \to G$. Denote $\ker(f : \pi_1(H_{\breve{\mathbb{Q}}_p})_I \to \pi_1(G_{\breve{\mathbb{Q}}_p})_I)$ by $\pi(H, G)$.*

Given $f : H \to G$ and $g : G \to G'$, we have, by definition, an inclusion $\pi(H, G) \subset \pi(H, G')$. It follows from the definition of (quasi-)parahoric subgroups that

Lemma 6.19. *To show that (G, X) is accessible to (G_1, X_1), it suffices to show that*

$$\pi(G^{\mathrm{der}}, G) \subset \pi(G^{\mathrm{der}}, G_1).$$

Proposition 6.20. *Fix a prime p. Let (G_2, X_2) be an abelian-type Shimura datum. We assume that there is always a Hodge-type lifting (G, X) of (G_2, X_2) such that*

(1) For any place v' of $\mathbb{E}'$ over a place v_2 of $\mathbb{E}_2$ over p, v' splits completely over v_2;
(2) (G, X) is accessible to (G_2, X_2) at p.

Then, for any cusp label representative $\Phi_2 \in \mathcal{CLR}(G_2, X_2)$ and any quasi-parahoric subgroup $K_{\Phi_2,p}$ as in Construction 6.16(1), the maps $\mathscr{S}_{K_{\Phi_2}} \to \mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}}$, $\mathscr{S}_{\overline{K}_{\Phi_2}} \to \mathscr{S}_{\overline{K}^{\mathrm{stb}}_{\Phi_2}}$ and $\mathscr{S}_{K_{\Phi_2,h}} \to \mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2,h}}$ are finite étale.

Proof. With these two assumptions, the equality $\pi^b(K_p^\circ)\pi^a(K_{2,p}) \cap G_2(\mathbb{Q}_p) = K_{2,p}$ holds. Indeed, it suffices to show that $\pi^b(K_p^\circ) \cap G_2(\mathbb{Q}_p) \subset K_{2,p}$. Since $\pi^b(G(\mathbb{Q}_p)) \cap G_2(\mathbb{Q}_p) \subset G_2^{\mathrm{der}}(\mathbb{Q}_p)$, we show that $\pi^b(K_p^\circ) \cap G_2^{\mathrm{der}}(\mathbb{Q}_p) \subset K_{2,p} \cap G_2^{\mathrm{der}}(\mathbb{Q}_p)$. Since the kernel of π^b is in the center of G^{der}, we know that if $x \in G(\mathbb{Q}_p)$ maps to $G'^{,\mathrm{der}}(\mathbb{Q}_p) = G_2^{\mathrm{der}}(\mathbb{Q}_p)$, then $x \in G^{\mathrm{der}}(\mathbb{Q}_p)$. Thus, $\pi^b(K_p^\circ \cap G^{\mathrm{der}}(\mathbb{Q}_p)) = \pi^b(K_p^\circ) \cap G'^{,\mathrm{der}}(\mathbb{Q}_p)$. Finally, we know from the assumption that $\pi^b(K_p^\circ \cap G^{\mathrm{der}}(\mathbb{Q}_p)) = \pi^b(K_p^\circ) \cap G'^{,\mathrm{der}}(\mathbb{Q}_p) \subset K_{2,p} \cap G_2^{\mathrm{der}}(\mathbb{Q}_p)$.

When this equality holds, for any quasi-parahoric $K_{2,p} \subset G_2(\mathbb{Q}_p)$, we can choose $K'_p \subset G'(\mathbb{Q}_p)$ such that $K'_p \cap G_2(\mathbb{Q}_p) = K_{2,p}$. Moreover, $\pi^{b,-1}(K'_p)$ are quasi-parahoric subgroups. See [Wu25, Lem. 4.57].

Now, $\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}}$ can be constructed by choosing the corresponding stabilizer quasi-parahoric subgroup $\widetilde{K}^{\mathrm{stb}}_{\Phi,p}$ of $ZP_\Phi(\mathbb{Q}_p)$, and $\mathscr{S}_{K^{\mathrm{min}}_{\Phi_2}}$ can be constructed from $\mathscr{S}_{\widetilde{K}_\Phi}$ such that $\widetilde{K}_{\Phi,p}$ is an open compact subgroup whose preimage in $G(\mathbb{Q}_p)$ contains the corresponding parahoric subgroups of $P_{\Phi_0^\alpha}(\mathbb{Q}_p)$. Then, by the statement that the quotient of each $\Delta_{\Phi_{\beta'},K'}(ZP_\Phi)$ is finite étale proved in [Wu25, Lem. 4.25], we are reduced to showing the corresponding statement for Hodge-type Shimura data. This follows from Proposition 4.9(2), noting that the assumption there is trivial by [Wu25, Cor. 1.9]. ☐

The crucial part is to show that

Proposition 6.21. *There always exists a Hodge-type lifting (G, X) such that the assumptions in Proposition 6.20 are true.*

Construction 6.22. For an abelian-type (G_2, X_2), we associate a Hodge-type (G, X) such that it satisfies (1) of Proposition 6.20; this can be done by the construction as in [Del79, Prop. 2.3.10] and [KPZ24] for any prime p. We now modify the Shimura datum to make it to satisfy (2) of Proposition 6.20.

Let $(T, h) \subset (G, X)$ be a special point with T a maximal torus. We first consider $G^{db} := G \times_{G^{ab}} G$ where the map $G \to G^{ab} = G/G^{\mathrm{der}}$ is the natural one. Denote $T^{ab} := T/(T \cap G^{\mathrm{der}})$. Define a refined construction $G^{rf} := G \times_{G^{ab}} T$.

Since (G, X) is of Hodge-type, we can decompose G to an almost-direct product $G = G^{\mathrm{der}} \cdot Z^c \cdot \mathbb{G}_m$, where Z^c is an $\mathbb{R}$-anisotropic torus. Then $G^{db} \cong (G^{\mathrm{der}} \times G^{\mathrm{der}}) \cdot Z^c \cdot \mathbb{G}_m \subset G \times G$.

In the second expression of the above line, $Z^c \cdot \mathbb{G}_m$ maps to $G \times G$ diagonally into $(Z^c \cdot \mathbb{G}_m) \times (Z^c \cdot \mathbb{G}_m)$. Alternatively, $G^{db} \cong ((G^{\mathrm{der}} \times G^{\mathrm{der}}) \times Z^c \cdot \mathbb{G}_m)/Z^c \cdot \mathbb{G}_m \cap G^{\mathrm{der}}$, where the last intersection maps to the first factor diagonally and to the second factor naturally.

Denote the corresponding Shimura data by (G^{db}, X^{db}) and (G^{rf}, X^{rf}), respectively. There is a natural map $(G^{rf}, X^{rf}) \to (G^{db}, X^{db})$. Then both of them are of Hodge type because they are both contained in the Hodge-type Shimura datum defined by $G \times_{\eta, \mathbb{G}_m, \eta} G$.

Proposition 6.23. *Proposition 6.21 is true for any prime p. Given (G_2, X_2) of abelian type, the accessible Hodge-type lifting is (G^{rf}, X^{rf}), as given by Construction 6.22.*

More precisely, $\pi(G^{rf,\mathrm{der}}, G^{rf})$ is trivial. In particular, the intersection of any parahoric subgroup $\breve{K}_p$ of $G^{rf}(\breve{\mathbb{Q}}_p)$ with $G^{rf,\mathrm{der}}(\breve{\mathbb{Q}}_p)$ is parahoric.

Proof. It suffices to show that $\pi(G^{rf,\mathrm{der}}, G^{rf})$ is trivial. We first consider $\pi(G^{db,\mathrm{der}}, G^{db})$. By Lemma 6.24 below, we have that

$$\pi(G^{db,\mathrm{der}}, G^{db}) = \mathrm{diag}_{\{1,2\}}\{\pi(G^{\mathrm{der}}, G)\},$$

where the index $\{1, 2\}$ labels the two factors of $G^{db,\mathrm{der}} = G^{\mathrm{der}} \times G^{\mathrm{der}}$.

Note that $G^{rf,\mathrm{der}} = G^{\mathrm{der}} \times \{1\} \subset G^{db,\mathrm{der}}$. Then $\pi(G^{\mathrm{der}} \times \{1\}, G^{db}) \subset \pi(G^{db,\mathrm{der}}, G^{db})$ is trivial. So $\pi(G^{rf,\mathrm{der}}, G^{rf}) \subset \pi(G^{rf,\mathrm{der}}, G^{db})$ is trivial. ☐

Lemma 6.24. *Let G be any reductive group, $\pi : G \to G^{ab}$, $G^{db} = G \times_{G^{ab}} G$. Then*

$$\pi(G^{db,\mathrm{der}}, G^{db}) = \mathrm{diag}_{\{1,2\}}\, \pi(G^{\mathrm{der}}, G) \subset \pi(G^{\mathrm{der}} \times G^{\mathrm{der}}, G \times G).$$

Proof. Consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
 & G^{\mathrm{der}} \times G^{\mathrm{der}} \arrow[r, equal] \arrow[d, hook] & G^{\mathrm{der}} \times G^{\mathrm{der}} \arrow[d, hook] & & \\
0 \arrow[r] & G^{db} \arrow[r, "i"] \arrow[d, two heads, "\pi"] & G \times G \arrow[r, "{(\pi,\pi^{-1})}"] \arrow[d, two heads] & G^{ab} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G^{ab} \arrow[r, "\Delta"] & G^{ab} \times G^{ab} \arrow[r, "{(\mathrm{id},\mathrm{id}^{-1})}"] & G^{ab} \arrow[r] & 0.
\end{tikzcd}$$

Take the long exact sequences of it,

$$\begin{tikzcd}
H_1(I, \pi_1(G^{ab})_I) \arrow[r, hook, "\Delta"] \arrow[d, "\Delta(\delta)"] & H_1(I, \pi_1(G^{ab})_I) \times H_1(I, \pi_1(G^{ab})_I) \arrow[d, "{(\delta,\delta)}"] \\
\pi_1(G^{\mathrm{der}})_I \times \pi_1(G^{\mathrm{der}})_I \arrow[r, dash, "="] \arrow[d] & \pi_1(G^{\mathrm{der}})_I \times \pi_1(G^{\mathrm{der}})_I \arrow[d] \\
\pi_1(G^{db})_I \arrow[r, "i"] & \pi_1(G)_I \times \pi_1(G)_I
\end{tikzcd}$$

From the diagram, we see that $\pi(G^{db,\mathrm{der}}, G^{db}) = \mathrm{im}\,\Delta(\delta)$, and $\pi(G^{\mathrm{der}} \times G^{\mathrm{der}}, G \times G) = \mathrm{im}\,\delta \times \delta$. It follows from definition that $\mathrm{im}\,\Delta(\delta) \subset \mathrm{im}\,\delta \times \delta$ is exactly $\mathrm{diag}_{\{1,2\}}\,\mathrm{im}\,\delta = \mathrm{diag}_{\{1,2\}}\,\pi(G^{\mathrm{der}}, G)$. ☐

Proof of Proposition 6.21. Now let (G, X) be any Hodge lifting of (G_2, X_2). Since $G^{db} \subset G \times_{\eta,\mathbb{G}_{\mathrm{m}},\eta} G$, where $\eta : G \to \mathbb{G}_{\mathrm{m}}$ is the similitude character (here η factors through G^{ab}), (G^{db}, X^{db}) is a Hodge-type Shimura datum. Therefore, (G^{rf}, X^{rf}) is a Hodge-lifting of (G_2, X_2) and satisfies the second condition of Proposition 6.20. ☐

Remark 6.25. *We remark that, in our construction, $G = G^{rf}$ might not be R-smooth when $p = 2$, and that Z_G might not be connected. Some supplementary examples are provided in a note [MW26].*

6.2.3. *Main theorem for boundary mixed Shimura varieties of abelian type.*

Theorem 6.26. *Let (G_2, X_2) be an abelian-type Shimura datum, and let $\mathcal{G}_2$ be any quasi-parahoric model of $G_{2,\mathbb{Q}_p}$. Then the construction above gives a family of canonical integral models*

$$\{\mathscr{S}_{K_{\Phi_2}}(P_{\Phi_2}, D_{\Phi_2})\}_{K^p_{\Phi_2}}$$

that is adapted with $P_{\Phi_2} \to G_{\Phi_2,h}$ and $G_2 \to G_2^c$. Here we can take $\{K^p_\Phi\}$ to be the collection of all neat open compact subgroups in $G_2(\mathbb{A}^p_f)$.

Proof. As the first step, we show that $\{\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}} := \mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi^b)\}_{K',p}$ is a canonical integral model adapted with $ZP_\Phi \to ZG_{\Phi,h} := ZP_\Phi/W_\Phi$ and $ZP_\Phi \to ZP_\Phi^c$ in the sense of Definition 4.6 and Definition 4.5, respectively.

Note that there is a morphism $\pi^c_{\widetilde{K}_\Phi^{\mathrm{stb}}} : \mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}} \to \mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb},c}}(ZP_\Phi^c, (ZP_\Phi(\mathbb{Q})D_\Phi^b)^c)$ by construction. Indeed, in Construction 6.16, we replace G' in (G', X_b) with $G'^{,c}$. Since $G' = G_2 \times^{G_2^{\mathrm{der}}} (G/C^{\mathrm{der}})$ and $G = G^c$, we see that $\pi^{b,c} : (G, X) \to (G'^{,c}, X_b^c)$ also satisfies the setup in Construction 6.14. The morphism $\pi^c_{\widetilde{K}_\Phi^{\mathrm{stb}}}$ is finite étale since we can write the map as

$$\coprod_{\beta \in I_{\Phi, K'^{,\mathrm{stb}}}} \mathscr{S}_{K_{\Phi_\beta}}/\Delta_{\Phi_\beta, K'^{,\mathrm{stb}}}(ZP_\Phi) \to \coprod_{\beta^c \in I_{\Phi^c, K'^{,\mathrm{stb},c}}} \mathscr{S}_{K_{\Phi_{\beta^c}}}/\Delta_{\Phi_{\beta^c}, K'^{,\mathrm{stb},c}}(ZP_\Phi^c),$$

and for each β mapping to β', there is a commutative diagram

$$\begin{tikzcd}
\mathscr{S}_{K_{\Phi_\beta}} \arrow[r] \arrow[d] & \mathscr{S}_{K_{\Phi_{\beta^c}}} \arrow[d] \\
\mathscr{S}_{K_{\Phi_\beta}}/\Delta_{\Phi_\beta, K'^{,\mathrm{stb}}}(ZP_\Phi) \arrow[r] & \mathscr{S}_{K_{\Phi_{\beta^c}}}/\Delta_{\Phi_{\beta^c}, K'^{,\mathrm{stb},c}}(ZP_\Phi^c),
\end{tikzcd}$$

such that all but the bottom arrow are already known to be finite étale.

We check that the models $\{\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}}\}_{K',p}$ are Pappas-Rapoport canonical integral models in the sense of Axiom 4.2. For (2), it follows from the construction and [Wu25, Lem. 4.25]. For (1), we choose any discrete valuation ring R with fraction field F. Choose any prime $l \neq p$. Choose a F-point x of $\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb},l}}$. Then it follows from Construction 6.16 (2) (and (3)) and the proof of [Wu25, Prop. 4.30] that the action of $\varprojlim_{K_l'} \Delta_{\Phi_\beta, K'^{,\mathrm{stb}}}(ZP_\Phi)$ on $\varprojlim_{K_l} \mathscr{S}_{K_{\Phi_\beta}}$ factors through a *finite* group. Thus, by Lemma 6.13, there is a finite field extension F' of F, such that x lifts to an F'-point $\widetilde{x}$ of some $\mathscr{S}_{K^l_{\Phi_\beta}}$.

Denote by R' the normalization of R in F'; this is still a *discrete* valuation ring. Then we can apply Lemma 6.13 in the Hodge-type case to get a unique extension of $\widetilde{x}$ to an R'-point $\widetilde{y}$ of $\mathscr{S}_{K^l_{\Phi_\beta}}$. This point maps to an R'-point y of $\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb},l}}$. The uniqueness of this lifting follows from separatedness. Now (a stronger version of) (1) follows because $R' \cap F = R$.

We now check (3) and (4). Denote by $(\mathscr{P}_\beta, \phi_\beta)$ the $\mathcal{P}_{\Phi_\beta}$-shtuka on $\mathscr{S}^\diamond_{K_{\Phi_\beta}}$ given by Proposition 6.11. We push out $\mathscr{P}_\beta$ via $\mathcal{P}_{\Phi_\beta} \to g_{\Phi_\beta} \mathcal{Z}\mathcal{P}_\Phi^c g_{\Phi_\beta}^{-1} \xrightarrow{\sim} \mathcal{Z}\mathcal{P}_\Phi^c$; we denote by $(\mathscr{Q}_\beta, \psi_\beta)$ the pushout. By checking over the generic fiber, the Δ_{Φ_β}-action lifts to an action on $(\mathscr{Q}_\beta, \psi_\beta)$. By étale descent, there is a $\mathcal{Z}\mathcal{P}_\Phi^c$-shtuka $(\widetilde{\mathscr{P}}, \widetilde{\phi})$ on $\mathscr{S}^\diamond_{\widetilde{K}_\Phi^{\mathrm{stb}}}$. This checks (3). For (4), it follows again from corresponding (4) in the Hodge-type case and [Wu25, Lem. 4.25] and from the dévissage of integral local Shimura varieties by [PR26, Prop. 5.3.1].

We can also verify Axiom 4.2 for $\{\mathscr{S}_{\widetilde{K}_{\Phi,h}^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi^b)\}$, $\{\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb},c}}(ZP_\Phi^c, (ZP_\Phi(\mathbb{Q})D_\Phi^b)^c)\}$ and $\{\mathscr{S}_{\widetilde{K}_{\Phi,h}^{\mathrm{stb},c}}(ZP_\Phi^c, (ZP_\Phi(\mathbb{Q})D_\Phi^b)^c)\}$ in exactly the same way. We now have finished the first step.

For the second step, we show that above properties for integral models associated with $(G', X_b, \widetilde{K}_\Phi^{\mathrm{stb}})$ can be transferred to the integral models associated with $(G', X_a, \widetilde{K}_\Phi^{\mathrm{stb}})$: All geometric properties required here can be checked over an étale cover. We base change to $\mathcal{O}^{ur}$ and descend the schemes to $\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi^a) \to \mathscr{S}_{\widetilde{K}_{\Phi,h}^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi^a)$ and $\mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D_\Phi^a) \to \mathscr{S}_{\widetilde{K}_\Phi^{\mathrm{stb}}}(ZP_\Phi^c, (ZP_\Phi(\mathbb{Q})D_\Phi^a)^c)$. The latter map is again finite étale by étale descent. Note that the Weil descent datum of shtukas extends from the generic fiber by [PR24, Cor. 2.7.10].

For the third step, we show that $\{\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}}\}$, $\{\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2,h}}\}$, $\{\mathscr{S}_{K^{\mathrm{stb},c}_{\Phi_2}}\}$, $\{\mathscr{S}_{K^{\mathrm{stb},c}_{\Phi_2,h}}\}$ satisfy Axiom 4.2 and has the desired properties. We apply Lemma 4.8(1) to the embedding of mixed Shimura data $(P_{\Phi_2}, D_{\Phi_2}) = (P_\Phi, D_\Phi) \to (ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)$, to verify axiom (3) (and also axiom (1), (2), as in the proof of Proposition 4.9 (1)). Note that, since the whole tower $\{\mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)\}_{K'^{,p}}$ is constructed, we can adjust prime-to-p levels so that the cited lemma can be applied. In fact, we adjust $K'^{,p}$ so that both $\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}} \to \mathscr{S}_{\widetilde{K}^{\mathrm{stb}}_\Phi}(ZP_\Phi, ZP_\Phi(\mathbb{Q})D^a_\Phi)$ and $\mathscr{S}_{K^{\mathrm{stb},c}_{\Phi_2}} \to \mathscr{S}_{\widetilde{K}^{\mathrm{stb},c}_\Phi}(ZP^c_\Phi, (ZP_\Phi(\mathbb{Q})D^a_\Phi)^c)$ are open and closed embeddings. We see from this that $\mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}} \to \mathscr{S}_{K^{\mathrm{stb},c}_{\Phi_2}}$ is finite étale for any $K^{\mathrm{stb},p}_{\Phi_2}$. In particular, axiom (4) follows easily, using the *dévissage* of integral local Shimura varieties ([PR26, Prop. 5.3.1]).

Now, we consider a general quasi-parahoric level $K_{2,p}$ and finish by applying Proposition 4.9(2) to $K_{2,p} \to K^{\mathrm{stb}}_{2,p}$. We just need to verify the condition highlighted there. We adopt the conventions in Construction 6.16. In fact, $\mathscr{S}_{K^c_{\Phi_2}} \to \mathscr{S}_{K^{\mathrm{stb},c}_{\Phi_2}}$ is finite étale by applying Proposition 4.9(2), whose condition is trivial for this case. Now, it suffices to show that $\mathscr{S}_{K_{\Phi_2}} \to \mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2}}$, $\mathscr{S}_{\overline{K}_{\Phi_2}} \to \mathscr{S}_{\overline{K}^{\mathrm{stb}}_{\Phi_2}}$ and $\mathscr{S}_{K_{\Phi_2,h}} \to \mathscr{S}_{K^{\mathrm{stb}}_{\Phi_2,h}}$ are finite étale. This is done by Proposition 6.21 and Proposition 6.20. □

7. WELL-POSITIONED SUBSCHEMES

Fix a Shimura datum (G, X) and a quasi-parahoric model $\mathcal{G}$ of $G_{\mathbb{Q}_p}$. For each $\Phi \in \mathcal{CLR}(G, X)$, we denote $G_\Phi = gG_{\mathbb{Q}_p}g^{-1}$, where $g = g_{\Phi,p} \in G(\mathbb{Q}_p)$. In this subsection, we generalize the results in [Box15], [LS18a] and [Mao25b]. In order to present the nature of those well-positioned subschemes, **we work under Assumption 5.26**.

Let us recall the definition of a well-positioned subscheme.

Definition/Proposition 7.1 ([LS18a, Def. 2.2.1 and Lem. 2.2.2])**.** *Let T be a locally noetherian scheme over $\mathcal{O}_E$. A locally closed subset (resp. subscheme) Y of $\mathscr{S}_{K,T} := (\mathscr{S}_K)_T$ is called* well positioned, *if, for each $\Phi \in \mathcal{CLR}(G, X)$, there exists a locally closed subset (resp. subscheme) $Y^\natural(\Phi) \subset \mathrm{Z}(\Phi)_T \overset{\sim}{\to} \mathcal{Z}([\Phi])_T$ such that for some (thus for all) cone decompositions Σ, and for each $\sigma \in \Sigma(\Phi)^+$, there are some (thus for all) open affine coverings $\mathfrak{W}$ of $\mathfrak{X}^\circ_\sigma$ satisfying the following property: the pullback of $Y^\natural(\Phi)$ along*

$$W^0_T \to \Delta^\circ_{\Phi,K}\backslash\Xi(\Phi)_T \to \Delta^\circ_{\Phi,K}\backslash C(\Phi)_T \to \Delta^\circ_{\Phi,K}\backslash \mathrm{Z}^\star(\Phi)_T \to \mathrm{Z}(\Phi)_T$$

coincides with the pullback of Y along $W^0_T \to \mathscr{S}_{K,T}$ as a subset (resp. subscheme). If this is the case, we say Y is well positioned with respect to $Y^\natural := \{Y^\natural(\Phi)\}_\Phi$, and $Y^\natural$ is associated with Y.

In practice, we usually take $T = s = \operatorname{Spec} k_E$ or $T = \bar{s} = \operatorname{Spec} \bar{k}_E$.

Remark 7.2. *Compared with* [LS18a, Def. 2.2.1 and Lem. 2.2.2], *we need to consider the $\Delta^\circ_{\Phi,K}$-action. Note that $\Delta^\circ_{\Phi,K}$ is trivial in the Hodge-type case, and thus was not considered in* [LS18a]. *With this modification, if we replace everything*

$$\mathrm{Z}^\star(\Phi),\ C(\Phi),\ \Xi(\Phi),\ \Xi(\Phi)(\sigma),\ \Xi(\Phi)_\sigma,\ \overline{\Xi}_\Sigma,\ \mathfrak{X}_{[\Phi,\sigma]},\ \mathfrak{X}^\circ_{[\Phi,\sigma]},\ \mathfrak{X}_\Sigma \tag{7.1}$$

by their quotients under the $\Delta^\circ_{\Phi,K}$-action, then the statements and proofs in [LS18a] *and* [Mao25b] *work without change.*

Remark 7.3. *The quotient map $\mathrm{Z}^\star(\Phi) \to \mathrm{Z}(\Phi)$ is finite étale by* [Wu25, Cor. 4.27]. *It follows that $\Delta^\circ_{\Phi,K}\backslash\mathrm{Z}^\star(\Phi) \to \mathrm{Z}(\Phi)$ is also finite étale. Under Assumption 5.26, $\Delta^\circ_{\Phi,K}\backslash C(\Phi) \to \mathrm{Z}(\Phi)$ is flat with geometrically reduced fibers, which verifies* [Mao25b, Assumption 2.7]. *In this situation, well-positioned subschemes enjoy more satisfactory properties.*

Definition 7.4 ([LS18a, Def. 2.3.1])**.** *Let* Y *be a locally closed subscheme of* $\mathscr{S}_{K,T}$. *Let* $\overline{Y}$ *be the closure of* Y *in* $\mathscr{S}_{K,T}$, $Y_0 = \overline{Y} \backslash Y$ *the complement of* Y *in* $\overline{Y}$. *Let* $\overline{Y}^{\min}$ *and* $Y_0^{\min}$ *be the closure of* $\overline{Y}$ *and* Y_0 *in* $\mathscr{S}_{K,T}^{\min}$ *respectively,* $Y^{\min}$ *be the complement of* $Y_0^{\min}$ *in* $\overline{Y}^{\min}$. *Similarly we define* $\overline{Y}^{\Sigma}$, Y_0^{Σ}, Y^{Σ}. *We call* $Y^{\min}$ *(resp.* Y^{Σ}*) the* partial minimal compactification *(resp. the* partial toroidal compactification *with respect to* Σ*) of* Y.

If $Y \subset \mathscr{S}_{K,T}$ is well positioned with respect to $Y^{\natural} := \{Y^{\natural}(\Phi)\}_{\Phi}$, then $\oint_{K,T}^{\Sigma} : \mathscr{S}_{K,T}^{\Sigma} \to \mathscr{S}_{K,T}^{\min}$ induces a morphism $\oint_{Y}^{\Sigma} : Y^{\Sigma} \to Y^{\min}$. Moreover, $(\oint_{K,T}^{\Sigma})^{-1}(Y^{\min}) = Y^{\Sigma}$, and the compactifications $Y^{\Sigma} \to Y^{\min}$ of Y satisfy Axiom 5.1 except for the flatness and normality of Y^{Σ}, of $Y^{\min}$, and of their boundary strata. See [LS18a, Thm. 2.3.2]. Note that, in Axiom 5.1, for any scheme (or formal scheme) in (7.1), we replace (?) by $Y^{\natural}_{(?)}$, which is defined as the pullback of $Y^{\natural}(\Phi)$ along $(?) \to \mathrm{Z}(\Phi)$. The identification $\mathrm{Z}(\Phi) \xrightarrow{\sim} \mathcal{Z}([\Phi])$ induces a canonical morphism $Y^{\natural}(\Phi) \to Y_{\mathcal{Z}([\Phi])} := (Y^{\min} \cap \mathcal{Z}([\Phi]))$, which induces a bijection on the underlying sets.

7.1. **Newton strata and central leaves.** We define Newton strata and central leaves following [PR24]. Let $x \in \mathscr{S}_K(G,X)(k)$. Pulling back the universal $\mathcal{G}^c$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ over $\mathscr{S}_K(G,X)^{\Diamond/}$, we get a $\mathcal{G}^c$-shtuka $(\mathscr{P}_x, \phi_{\mathscr{P}_x}) := x^*(\mathscr{P}, \phi_{\mathscr{P}})$ over $\mathrm{Spec}(k)$, which is associated with a $\mathcal{G}^c$-torsor on $\mathrm{Spec}\, W(k)$ together with an isomorphism $\phi_{\mathscr{P}_x} : \phi^*(\mathscr{P}_x)[1/p] \xrightarrow{\sim} \mathscr{P}_x[1/p]$. A choice of trivialization of $\mathscr{P}_x$ defines an element $b_x \in G^c(\breve{\mathbb{Q}}_p)$; a different choice of trivialization gives $\mathcal{G}^c(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugation of b_x. Let $C(\mathcal{G}^c) = G^c(\breve{\mathbb{Q}}_p)/\mathcal{G}^c(\breve{\mathbb{Z}}_p)_\sigma$ and $B(G^c) = G^c(\breve{\mathbb{Q}}_p)/G^c(\breve{\mathbb{Q}}_p)_\sigma$; we have

$$\Upsilon_K(k) : \mathscr{S}_K(G,X)(k) \to C(\mathcal{G}^c), \quad \delta_K(k) : \mathscr{S}_K(G,X)(k) \to B(G^c). \tag{7.2}$$

Since $(\mathscr{P}, \phi_{\mathscr{P}})$ is bounded by μ^c, the σ-conjugation class of b_x sits inside $B(G^c, \{\mu^{c,-1}\}) \subset B(G^c)$.

One can define Υ_K, δ_K globally, without referring to k-points. In fact, in subsection 1.3.3, we recall a natural projection $\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1} \to G^c\text{-}\mathrm{Isoc}_{\mu^{c,-1}}$. By construction, $\Upsilon_K(k)$ and $\delta_K(k)$ in (7.2) are the k-points of $\mathscr{S}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1}$ and $\mathscr{S}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to G^c\text{-}\mathrm{Isoc}_{\mu^{c,-1}}$, respectively. Note that perfection does not change the underlying topological space, so we have a morphism

$$\delta_K : \mathscr{S}_K(G,X)_{\bar{s}} \to B(G^c, \{\mu^{c,-1}\})$$

whose k-points are $\delta_K(k)$. Given $[b] \in B(G^c, \{\mu^{c,-1}\})$ with $b \in G^c(\breve{\mathbb{Q}}_p)$, let $\mathcal{N}^{[b]} \subset \mathscr{S}_K(G,X)_{\bar{s}}$ be the preimage $\delta_K^{-1}([b])$; we call it the *Newton stratum* associated with $[b]$. [RR96, Thm. 3.6] says that $\mathscr{S}_K(G,X)_{\bar{s}} \to B(G^c, \{\mu^{c,-1}\})$ is semi-lower continuous; therefore, Newton strata are locally closed. We endow them with the unique reduced subscheme structures.

Now we define Υ_K globally. Given $[[b]] \in C(\mathcal{G}^c)$ with $b \in G^c(\breve{\mathbb{Q}}_p)$, let $\mathcal{C}^{[[b]]}(k) \subset \mathscr{S}_K(G,X)(k)$ (resp. $\mathcal{N}^{[b]}(k) \subset \mathscr{S}_K(G,X)(k)$) be the preimage $\Upsilon_K(k)^{-1}([[b]])$ (resp. $\delta_K(k)^{-1}([b])$).

Lemma 7.5. $\mathcal{C}^{[[b]]}(k) \subset \mathcal{N}^{[b]}(k)$ *is closed.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathscr{S}_{K^\circ}(G,X)^{\Diamond/} \arrow[r, "g"] \arrow[dr] & \mathscr{S}_K(G,X)'^{\Diamond/} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{S}_K(G,X)^{\Diamond/} \arrow[d] \\
 & \mathrm{Sht}_{\mathcal{G}^{\circ,c},\mu^c,\delta=1} \arrow[r] & \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}.
\end{tikzcd} \tag{7.3}$$

where $\mathscr{S}_K(G,X)'^{\Diamond/}$ is the fiber product, and $\mathscr{S}_{K^\circ}(G,X)$ is the relative normalization of $\mathscr{S}_K(G,X)$ in $\mathrm{Sh}_{K^\circ}(G,X)$. Since $\mathrm{Sht}_{\mathcal{G}^{c,\circ},\mu^c,\delta=1} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c,\delta=1}$ is a finite étale torsor under the abelian group $\pi_0(\mathcal{G}^c)^{\phi}$, $\mathscr{S}_K(G,X)'^{\Diamond/}$ is represented by a normal flat scheme $\mathscr{S}_K(G,X)'$, and $\mathscr{S}_K(G,X)' \to \mathscr{S}_K(G,X)$ is a finite étale torsor under $\pi_0(\mathcal{G}^c)^{\phi}$, thanks to [DvHKZ26, Prop. 2.3.1]. We take the special fiber of the diagram (7.3), and denote the Newton strata and central leaves in $\mathscr{S}_K(G,X)'(k)$ by

$\mathcal{C}^{[[b]]}(k)$ and $\mathcal{N}^{[b]}(k)$, respectively, using $\mathrm{Sht}^{W}_{\mathcal{G}^{c,\circ},\mu^c,\delta=1}$. Since $\mathcal{N}^{[b]}(k)$ is locally closed in $\mathscr{S}_K(G,X)'(k)$ by [RR96], we claim that $\mathcal{C}^{[[b]]}(k)$ is closed in $\mathcal{N}^{[b]}(k)$. If this is true, by using torsors, a central leaf in $\mathscr{S}_K(G,X)(k)$ is the image of some $\mathcal{C}^{[[b]]}(k)$ under f, and its preimage under f is a topologically disjoint union of some $\mathcal{C}^{[[b_i]]}(k)$; then the lemma follows from the fact that f is finite étale.

Let $x \in \mathcal{C}^{[[b]]}(k)$. In [HK25, §2.14], given a perfect scheme S and a $\mathcal{G}^c$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ on S, the authors define

$$\mathcal{C}^{[[b]]}_{(\mathscr{P},\phi_{\mathscr{P}})} := \left\{ s \in S |\ \bar{s}^*(\mathscr{P}, \phi_{\mathscr{P}}) \cong (\mathscr{P}_x, \phi_{\mathscr{P}_x})|_{\overline{\kappa(s)}} \right\},$$

and similarly define $\mathcal{N}^{[b]}_{(\mathscr{P},\phi_{\mathscr{P}})}$ using isocrystals. [HK25, Prop. 2.15(3)] says that $\mathcal{C}^{[[b]]}_{(\mathscr{P},\phi_{\mathscr{P}})} \subset \mathcal{N}^{[b]}_{(\mathscr{P},\phi_{\mathscr{P}})}$ is closed. Now let S be the perfection of $\mathcal{N}^{[b]}$; then $\mathcal{N}^{[b]}_{(\mathscr{P},\phi_{\mathscr{P}})}$ is the whole space $\mathcal{N}^{[b],\mathrm{perf}}$. Thus, $\mathcal{C}^{[[b]]}(k) = \mathcal{C}^{[[b]]}_{(\mathscr{P},\phi_{\mathscr{P}})}(k) \subset \mathcal{N}^{[b]}(k)$ is closed. ☐

By some standard arguments on Jacobson schemes, (e.g., [Mao25a, §3.2]), there exists a unique closed subscheme $\mathcal{C}^{[[b]]} \subset \mathcal{N}^{[b]}$ with induced reduced subscheme structure such that the set of k-points of $\mathcal{C}^{[[b]]}$ is $\mathcal{C}^{[[b]]}(k)$. Finally, let

$$C(\mathcal{G}^c, \{\mu^{c,-1}\}) := \mathrm{Sht}^{W}_{\mathcal{G}^c,\mu^c,\delta=1}(k),$$

then we obtain a globally defined Υ_K, which is determined by its values on k-points $\Upsilon_K(k)$:

$$\Upsilon_K : \mathscr{S}_K(G,X)_{\bar{s}} \to C(\mathcal{G}^c, \{\mu^{c,-1}\}),$$

and we call the fibers *central leaves*. In the proof of Lemma 7.5, fix a k-point $s \in \mathcal{C}^{[[b]]}$, we see that

$$\mathcal{C}^{[[b]]} = \left\{ x \in \mathscr{S}_K(G,X)_{\bar{s}} |\ (\mathscr{P}_s, \phi_{\mathscr{P}_s})|_{\overline{\kappa(x)}} \cong (\mathscr{P}_x, \phi_{\mathscr{P}_x})|_{\overline{\kappa(x)}} \right\}.$$

Remark 7.6. *The above arguments and definitions also apply to $(G^*_{\Phi,h}, \mathcal{G}^*_{\Phi,h})$. The central leaves on $\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})_{\bar{s}}$, defined using $\mathrm{Sht}^{W}_{\mathcal{G}^c_{\Phi,h},\mu^c_{\Phi,h},\delta=1}$, are finer than those defined using $\mathrm{Sht}^{W}_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1}$. Nevertheless, the index set of Newton strata (and of KR strata, of EKOR strata, recalled in the next subsection) depends only on the adjoint group, so there is no difference in whether one uses $\mathcal{G}^c_{\Phi,h}$ or $\mathcal{G}^*_{\Phi,h}$.*

Proposition 7.7. *Newton strata are well positioned. Moreover, let $\mathcal{N}^{[b]}$ be a Newton stratum on $\mathscr{S}_K(G,X)_{\bar{s}}$ with some $[b] \in B(G^c, \{\mu^{c,-1}\})$. Then, for each $\Phi \in \mathcal{CLR}(G,X)$, $(\mathcal{N}^{[b]})^{\natural}_{\mathrm{Z}^\star(\Phi)}$ is either empty or a Newton stratum $\mathcal{N}^{[b_{\Phi,h}]}$ on $\mathrm{Z}^\star(\Phi)_{\bar{s}} \cong \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})_{\bar{s}}$, for some $[b_{\Phi,h}] \in B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$. The relation between $[b]$ and $[b_{\Phi,h}]$ is given in the proof.*

Proof. We pass the commutative diagram (5.21) to isocrystals:

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \arrow[d] & W^{0,\mathrm{perf}}_{\bar{s}} \arrow[l] \arrow[r] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)^{\mathrm{perf}}_{\bar{s}} \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})^{\mathrm{perf}}_{\bar{s}} \arrow[d] \\
G^c\text{-}\mathrm{Isoc}_{\mu^{c,-1}} & & P^*_\Phi\text{-}\mathrm{Isoc}_{\mu^{*,-1}_\Phi} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & G^*_{\Phi,h}\text{-}\mathrm{Isoc}_{\mu^{*,-1}_{\Phi,h}} .
\end{tikzcd}$$

We further take the underlying topological spaces and get the commutative diagram

$$\begin{tikzcd}
\mathscr{S}_K(G,X)_{\bar{s}} \arrow[d] & W^{0}_{\bar{s}} \arrow[l] \arrow[r] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(P_\Phi, D_\Phi)_{\bar{s}} \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})_{\bar{s}} \arrow[d] \\
B(G^c, \{\mu^{c,-1}\}) & & B(P^*_\Phi, \{\mu^{*,-1}_\Phi\}) \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\}).
\end{tikzcd}$$

Let $[b] \in B(G^c, \{\mu^{c,-1}\})$. If $\mathcal{N}^{[b]} \subset \mathscr{S}_K(G,X)_{\bar{s}}$ is non-empty over $W^0_{\bar{s}}$, then $[b]$ is in the image of $\mathrm{Int}(g_\Phi^{-1}) : B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) \to B(G^c, \{\mu^{c,-1}\})$ for some $[b_\Phi] \in B(P_\Phi^*, \{\mu_\Phi^{*,-1}\})$.

On the other hand, let $[b_{\Phi,h}] \in B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$ be the image of $[b_\Phi]$. Since $B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) \xrightarrow{\sim} B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$ is a bijection, the preimage of $\mathcal{N}^{[b]}$ over $W^0_{\bar{s}}$ is the finite union of preimages of those Newton strata $\mathcal{N}^{[b_{\Phi,h}]}$ on $\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})_{\bar{s}}$, where $[b_{\Phi,h}]$ are the preimages of $[b] \in B(G^c, \{\mu^{c,-1}\})$ under $B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\}) \xrightarrow{\sim} B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) \to B(G^c, \{\mu^{c,-1}\})$.

We further claim that $B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) \to B(G^c, \{\mu^{c,-1}\})$ is injective; then there is a unique $[b_{\Phi,h}]$ associated with $[b]$. By [HNY24, Lem. 2.1], we have injectivity:

$$B(L_\Phi^*, \{\mu_\Phi^{*,-1}\}) = B(Q_\Phi^*, \{\mu_\Phi^{*,-1}\}) \to B(G_\Phi^c, \{\mu_\Phi^{c,-1}\}) = B(G^c, \{\mu^{c,-1}\}).$$

Also, note that $B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) \to B(Q_\Phi^*, \{\mu_\Phi^{*,-1}\})$ is bijective:

$$B(P_\Phi^*, \{\mu_\Phi^{*,-1}\}) = B(G^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\}) = B(G^{\mathrm{ad}}_{\Phi,h}, \{\mu^{\mathrm{ad},-1}_{\Phi,h}\}),$$

$$B(Q_\Phi^*, \{\mu_\Phi^{*,-1}\}) = B(L_\Phi^*, \{\mu^{*,-1}_{\Phi,h}\}) = B(L^{\mathrm{ad}}_\Phi, \{\mu^{\mathrm{ad},-1}_{\Phi,h}\}),$$

and

$$B(L^{\mathrm{ad}}_\Phi, \{\mu^{\mathrm{ad},-1}_{\Phi,h}\}) = B(G^{\mathrm{ad}}_{\Phi,h}, \{\mu^{\mathrm{ad},-1}_{\Phi,h}\}) \times B(G^{\mathrm{ad}}_{\Phi,l}, \{\mathrm{id}\}) = B(G^{\mathrm{ad}}_{\Phi,h}, \{\mu^{\mathrm{ad},-1}_{\Phi,h}\}).$$

Finally, to show that Newton strata are well positioned, we pass to $\mathrm{Z}(\Phi) = \Delta_{\Phi,K} \backslash \mathrm{Z}^\star(\Phi)$. It suffices to show that $\Delta_{\Phi,K}$ stabilizes each Newton stratum; then there is a unique subscheme $(\mathcal{N}^{[b]})^\natural \subset \mathrm{Z}(\Phi)_{\bar{s}}$ whose preimage in $\mathrm{Z}^\star(\Phi)_{\bar{s}}$ is $(\mathcal{N}^{[b]})^\natural_{\mathrm{Z}^\star(\Phi)} = \mathcal{N}^{[b_{\Phi,h}]}$. Such $(\mathcal{N}^{[b]})^\natural \subset \mathrm{Z}(\Phi)_{\bar{s}}$ is automatically locally closed if it exists; see [LS18a, Lem. 2.3.10]. Consider the $\Delta_{\Phi,K}$-action on Newton strata; now we apply Proposition 5.33. ☐

Remark 7.8. *As explained in* [Mao25b, Lem. 2.16, 2.20, Prop. 2.30]*, the connected components and closures of Newton strata, as well as $\mathcal{N}^{\leq [b]}$ for any $[b] \in B(G^c, \{\mu^{c,-1}\})$, are all well-positioned. The same applies to central leaves, KR strata, and EKOR strata proved in later sections, and we do not repeat the arguments there.*

The following proof was explained to us by Sian Nie.

Lemma 7.9. *Let G be any reductive group over $\mathbb{Q}_p$, M be any proper Levi subgroup of G, μ be a cocharacter of M that is non-central in G. The image of $B(M, \{\mu\}) \to B(G, \{\mu\})$ does not contain the basic element of $B(G, \{\mu\})$.*

Proof. Given $[b_1] \leq [b_2]$ in $B(M, \{\mu\})$, then $[b_1] \leq [b_2]$ in $B(G, \{\mu\})$. Since basic elements are minimal, it suffices to show that given the basic element $[b] \in B(M, \{\mu\})$, its image in $B(G, \{\mu\})$ can not be basic.

We fix a pinning and assume μ is dominant in G. Let $\mu^\diamond$ be the σ-average of μ, recall that

$$B(M, \{\mu\}) = \{[b] \in B(M) |\ \kappa([b]) \leq \kappa(\mu), \nu_b \leq \mu^\diamond\}$$

Let $[b] \in B(M, \{\mu\})$ be basic, then $\nu_b = \mu^\diamond - h$ is central in M, where h is a non-negative linear combination of positive coroots of M. Recall that $[b]$ is basic in $B(G, \{\mu\})$ if and only if ν_b is central in G if and only if $\langle \alpha, \nu_b \rangle = 0$ for all simple roots in G. For those simple roots α not in M, $\langle \alpha, \mu^\diamond \rangle \geq 0$ since $\mu^\diamond$ is dominant in G, and $\langle \alpha, h \rangle \leq 0$ by the Cartan matrix, then

$$\langle \alpha, \nu_b \rangle = \langle \alpha, \mu^\diamond \rangle - \langle \alpha, h \rangle \geq 0,$$

it takes equality if and only if $\langle \alpha, \mu^\diamond \rangle = \langle \alpha, h \rangle = 0$. This implies that $h = 0$. Since $\nu_b = \mu^\diamond$ is central in M, then $\mu^\diamond$ is central in G, we get a contradiction. ☐

Corollary 7.10. *Basic Newton stratum has no boundary.*

Proof. As in the proof of Proposition 7.7, it suffices to show that the basic element $[b_0] \in B(G^c, \{\mu^{c,-1}\})$ does not come from any Levi subgroup; this is proved in Lemma 7.9. Note that the Hodge cocharacter μ is central in G if and only if $G = T$ is a torus, and the Shimura variety associated with $(T, \{h\})$ has no boundary. ☐

Definition 7.11. *Let Y be a scheme and $\{Y_i\}_{i\in I}$ be subschemes of Y. We say that Y is a **topologically disjoint union** of $\{Y_i\}_{i\in I}$ if $Y = \bigsqcup_{i\in I} Y_i$ and all $Y_i \subset Y$ are open and closed.*

Proposition 7.12. *Central leaves are well positioned. Moreover, let $\mathcal{C}^{[[b]]}$ be a central leaf on $\mathscr{S}_K(G,X)_{\bar{s}}$ with some $[[b]] \in C(\mathcal{G}^c, \{\mu^{c,-1}\})$. Then, for each $\Phi \in \mathcal{CLR}(G,X)$, $(\mathcal{C}^{[[b]]})^{\natural}_{\mathrm{Z}^\star(\Phi)}$ is either empty or a topologically disjoint union of central leaves $\mathcal{C}^{[[b_{\Phi,h}]]}$ on $\mathrm{Z}^\star(\Phi)_{\bar{s}} \cong \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})_{\bar{s}}$, for some collection of $[[b_{\Phi,h}]] \in C(\mathcal{G}^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$. The relation between $[[b]]$ and the collection of $[[b_{\Phi,h}]]$ is given in the proof.*

Proof. Let $[[b]] \in C(\mathcal{G}^c, \{\mu^{c,-1}\})$. If $\mathcal{C}^{[[b]]} \subset \mathscr{S}_K(G,X)_{\bar{s}}$ is non-empty over $W^0_{\bar{s}}$, then $[[b]]$ is in the image of some $[[b_\Phi]]$ under $\mathrm{Int}(g_\Phi^{-1}) : C(\mathcal{P}^*_\Phi, \{\mu^{*,-1}_\Phi\}) \to C(\mathcal{G}^c, \{\mu^{c,-1}\})$; here $C(\mathcal{P}^*_\Phi, \{\mu^{*,-1}_\Phi\}) := \mathrm{Sht}^W_{\mathcal{P}^*_\Phi, \mu^*_\Phi}(k)$. We let $[[b_{\Phi,h}]] \in C(\mathcal{G}^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$ be the image of $[[b_\Phi]]$.

We claim that there is at most one $[[b_\Phi]] \in C(\mathcal{P}^*_\Phi, \{\mu^{*,-1}_\Phi\})$ in the image of $\mathscr{S}_{K_\Phi}(k) \to C(\mathcal{P}^*_\Phi, \{\mu^{*,-1}_\Phi\})$ that has fixed image $[[b_{\Phi,h}]] \in C(\mathcal{G}^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\})$. If this is the case, the preimage of $\mathcal{C}^{[[b]]}$ over $W^0_{\bar{s}}$ is the finite union of preimages of those central leaves $\mathcal{C}^{[[b_{\Phi,h}]]}$ on $\mathscr{S}_{K_{\Phi,h},\bar{s}}$, where $[[b_{\Phi,h}]]$ are the preimages of $[[b]] \in C(\mathcal{G}^c, \{\mu^{c,-1}\})$ under

$$C(\mathcal{G}^*_{\Phi,h}, \{\mu^{*,-1}_{\Phi,h}\}) \leftarrow C(\mathcal{P}^*_\Phi, \{\mu^{*,-1}_\Phi\}) \overset{\mathrm{Int}(g_\Phi^{-1})}{\to} C(\mathcal{G}^c, \{\mu^{c,-1}\}).$$

Let us show the claim. We first work with abelian scheme torsor part. Let $x \in \mathcal{C}^{[[b_{\Phi,h}]]}(k) \subset \mathscr{S}_{K_{\Phi,h}}(k)$, $y_1, y_2 \in \mathscr{S}_{\overline{K}_\Phi}(k)$ be in the preimage of x. Since $\mathscr{S}_{\overline{K}_\Phi} \to \mathscr{S}_{K_{\Phi,h}}$ is a torsor under the abelian scheme $\mathcal{A}_K(\Phi) \to \mathscr{S}_{K_{\Phi,h}}$, there exists $\gamma \in \mathcal{A}_K(\Phi)(k)$ with image x such that $\gamma y_1 = y_2$. We could lift y_1, y_2, x, γ to some $\mathcal{O}_F$-points $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{x}$, $\tilde{\gamma}$ respectively, where F is a finite extension of $\breve{\mathbb{Q}}_p$, such that $\tilde{\gamma}\tilde{y}_1 = \tilde{y}_2$ with image $\tilde{x}$. By Lemma 3.34 and [PR24, Corollary 2.7.10], there exists an isomorphism of $\overline{\mathcal{P}}^*_\Phi$-shtukas $(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ and $\tilde{\gamma}^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ over $\mathscr{S}_{\overline{K}_\Phi, \tilde{x}}$, thus there exists an isomorphism of $\overline{\mathcal{P}}^*_\Phi$-shtukas $y_1^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ and $y_2^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$. Next, we show that given different $x, x' \in \mathcal{C}^{[[b_{\Phi,h}]]}(k)$, one can take some points in the fiber $y, y' \in \mathscr{S}_{\overline{K}_\Phi}(k)$ respectively such that $y^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ is isomorphic to $y'^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$. Étale locally $T \to \mathscr{S}_{K_{\Phi,h}}$, there is a section $T \to \mathscr{S}_{\overline{K}_\Phi, T} \cong \mathcal{A}_K(\Phi)_T$, we can take it to be the zero section of the abelian scheme, such that its generic fiber $T_\eta \to \mathcal{A}_K(\Phi)_{T_\eta}$ coincides with the base change of the zero section $\mathrm{Sh}_{K_{\Phi,h}} \to \mathrm{Sh}_{K_{\Phi,V} \rtimes K_{\Phi,h}}$. This can be done using the group structure of $\mathcal{A}_K(\Phi)_T$. By Lemma 3.19 and [PR24, Cor. 2.7.10], we have a unique extension of morphism of shtukas:

$$\begin{tikzcd}
T^{\Diamond/} \arrow[r] \arrow[d] & \mathscr{S}^{\Diamond/}_{\overline{K}_\Phi, T} \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}^*_{\Phi,h}, \mu^*_{\Phi,h}} \arrow[r] & \mathrm{Sht}_{\overline{\mathcal{P}}^*_\Phi, \bar{\mu}^*_\Phi}.
\end{tikzcd}$$

In particular, for any $x \in \mathcal{C}^{[[b_{\Phi,h}]]}(k)$, we can find a preimage $y \in \mathscr{S}_{\overline{K}_\Phi}(k)$ such that the $\overline{\mathcal{P}}^*_\Phi$-shtuka $y^*(\overline{\mathscr{P}}_\Phi, \phi_{\overline{\mathscr{P}}_\Phi})$ is isomorphic to the pushforward of the $\mathcal{G}^*_{\Phi,h}$-shtuka $x^*(\overline{\mathscr{P}}_{\Phi,h}, \phi_{\overline{\mathscr{P}}_{\Phi,h}})$. This finishes the claim for the abelian scheme torsor part. Similar arguments can be made for the torus torsor part $\mathscr{S}_{K_\Phi} \to \mathscr{S}_{\overline{K}_\Phi}$, with Lemma 3.34 replaced by Lemma 3.29.

To show central leaves are well positioned, we need to pass to $\mathrm{Z}(\Phi) = \Delta_{\Phi,K} \backslash \mathrm{Z}^\star(\Phi)$ and show that $\Delta_{\Phi,K}$ stabilizes each central leaf. See the last paragraph of the proof of Proposition 7.7.

Finally, to show *topological disjointness*, note that central leaves are closed in Newton strata; this follows from Proposition 7.7 and the fact that if $Y_1 \subset Y_2$ is a closed embedding of well-positioned subschemes, then $Y^{\natural}_{1,\mathrm{Z}^\star(\Phi)} \subset Y^{\natural}_{2,\mathrm{Z}^\star(\Phi)}$ are closed embeddings for all $\Phi \in \mathcal{CLR}(G,X)$. See [Mao25b, Lem. 2.17, 2.18, 2.19]. ☐

7.1.1. *Toroidal compactifications.* Recall that we have globally defined shtukas on the special fiber of the integral model of the toroidal compactification by Proposition 5.29 and Corollary 5.30:

$$\mathscr{S}^{\Sigma}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1} \to G^c\text{-}\mathrm{Isoc}_{\mu^{c,-1}}, \tag{7.4}$$

which induces

$$\Upsilon^{\Sigma}_K(k) : \mathscr{S}^{\Sigma}_K(G,X)(k) \to C(\mathcal{G}^c,\{\mu^{c,-1}\}), \quad \delta^{\Sigma}_K : \mathscr{S}^{\Sigma}_K(G,X) \to B(G^c,\{\mu^{c,-1}\}).$$

Proposition 7.13. *Let $[b] \in B(G^c,\{\mu^{c,-1}\})$, and let $\mathcal{N}^{[b]} \subset \mathscr{S}_K(G,X)_{\bar{s}}$ be the Newton stratum. Then its partial toroidal compactification $(\mathcal{N}^{[b]})^{\Sigma}$ (see* [LS18a, Definition 2.3.1]*) is the fiber $\delta^{\Sigma,-1}_K([b])$.*

Proof. By restricting the diagram (5.18) to the special fiber, we have a commutative diagram:

$$\begin{tikzcd}
\mathscr{S}^{\Sigma}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \arrow[d] & W^{\mathrm{perf}}_{\bar{s}} \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)(\sigma)^{\mathrm{perf}}_{\bar{s}} \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h})^{\mathrm{perf}}_{\bar{s}} \arrow[d] \\
\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c,\delta=1} \arrow[d] & & \mathrm{Sht}^W_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] \arrow[d] & \mathrm{Sht}^W_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1} \arrow[d] \\
G^c\text{-}\mathrm{Isoc}_{\mu^{c,-1}} & & P^*_\Phi\text{-}\mathrm{Isoc}_{\mu^{*,-1}_\Phi} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & G^*_{\Phi,h}\text{-}\mathrm{Isoc}_{\mu^{*,-1}_{\Phi,h}} .
\end{tikzcd} \tag{7.5}$$

By Proposition 7.7, $\mathcal{N}^{[b]}$ is well positioned, and its preimage in $W^0_{\bar{s}}$ is the preimage of some $(\mathcal{N}^{[b]})^{\natural}_{\mathrm{Z}^\star(\Phi)} \subset \mathrm{Z}^\star(\Phi)_{\bar{s}} = \mathscr{S}_{K_{\Phi,h},\bar{s}}$. The preimage of its partial toroidal compactification $(\mathcal{N}^{[b]})^{\Sigma}$ in $W_{\bar{s}}$ should also be the preimage of the same $(\mathcal{N}^{[b]})^{\natural}_{\mathrm{Z}^\star(\Phi)}$; thus $(\mathcal{N}^{[b]})^{\Sigma}$ is contained in the fiber $(\delta^{\Sigma}_K)^{-1}([b])$. On the other hand, Newton strata form a (weak) stratification $\mathscr{S}_{K,\bar{s}} = \coprod_{[b]} \mathcal{N}^{[b]}$; thus $\mathscr{S}^{\Sigma}_K(G,X)_{\bar{s}} = \coprod_{[b]} (\mathcal{N}^{[b]})^{\Sigma}$ (see [Mao25b, Lem. 2.20]). This forces $(\mathcal{N}^{[b]})^{\Sigma} = (\delta^{\Sigma}_K)^{-1}([b])$. ☐

Proposition 7.14. *Let $[[b]] \in C(\mathcal{G}^c,\{\mu^{c,-1}\})$, and let $\mathcal{C}^{[[b]]} \subset \mathscr{S}_K(G,X)_{\bar{s}}$ be the central leaf. Then its partial toroidal compactification $(\mathcal{C}^{[[b]]})^{\Sigma}$ has the set of k-points $\Upsilon^{\Sigma}_K(k)^{-1}([[b]])$. In particular, we can upgrade $\Upsilon^{\Sigma}_K(k)$ to $\Upsilon^{\Sigma}_K : \mathscr{S}^{\Sigma}_K(G,X)_{\bar{s}} \to C(\mathcal{G}^c,\{\mu^{c,-1}\})$, and $(\mathcal{C}^{[[b]]})^{\Sigma} = \Upsilon^{\Sigma,-1}_K([[b]])$.*

Proof. We cannot directly apply the proof of Proposition 7.13, since

$$C(\mathcal{P}^*_\Phi,\{\mu^{*,-1}_\Phi\}) := \mathrm{Sht}^W_{\mathcal{P}^*_\Phi,\mu^*_\Phi,\delta=1}(k) \to C(\mathcal{G}^*_{\Phi,h},\{\mu^{*,-1}_{\Phi,h}\}) := \mathrm{Sht}^W_{\mathcal{G}^*_{\Phi,h},\mu^*_{\Phi,h},\delta=1}(k) \tag{7.6}$$

is not injective.

Taking the set of k-points of the diagram (7.5), we have

$$\begin{tikzcd}
\mathscr{S}^{\Sigma}_K(G,X)(k) \arrow[d] & W(k) \arrow[l] \arrow[r] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)(\sigma)(k) \arrow[r] \arrow[d] & \Delta^{\circ}_{\Phi,K}\backslash \mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h})(k) \arrow[d] \\
C(\mathcal{G}^c,\{\mu^{c,-1}\}) & & C(\mathcal{P}^*_\Phi,\{\mu^{*,-1}_\Phi\}) \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & C(\mathcal{G}^*_{\Phi,h},\{\mu^{*,-1}_{\Phi,h}\}).
\end{tikzcd}$$

By Proposition 7.12, $\mathcal{C}^{[[b]]}(k)$ is well positioned, and its preimage in $W(k)$ is the preimage of a union of central leaves $(\mathcal{C}^{[[b]]})^{\natural}_{\mathrm{Z}^\star(\Phi)}(k) \subset \mathrm{Z}^\star(\Phi)(k) = \mathscr{S}_{K_{\Phi,h}}(k)$. The preimage of $(\mathcal{C}^{[[b]]})^{\Sigma}(k)$ in $W(k)$ should also be the preimage of $(\mathcal{C}^{[[b]]})^{\natural}_{\mathrm{Z}^\star(\Phi)}(k)$. Let $\mathcal{C}^{[[b_{\Phi,h}]]} \subset (\mathcal{C}^{[[b]]})^{\natural}_{\mathrm{Z}^\star(\Phi)}$ be a central leaf. The arguments in the proof of Proposition 7.12 show that the preimage of $\mathcal{C}^{[[b_\Phi,h]]}(k)$ in $\mathscr{S}_{K_\Phi}(k)$ is a central leaf $\mathcal{C}^{[[b_\Phi]]}(k)$ with a unique $[[b_\Phi]] \in C(\mathcal{P}^c_\Phi, \{\mu^{c,-1}_\Phi\})$.

Let us denote by $\mathcal{C}^{[[b_\Phi]]}(\sigma)(k)$ (resp. $\mathcal{N}^{[b_\Phi]}(\sigma)(k)$) the preimage of $\mathcal{C}^{[[b_\Phi,h]]}(k)$ (resp. $\mathcal{N}^{[b_\Phi,h]}(\sigma)(k)$) in $\mathscr{S}_{K_\Phi}(\sigma)(k)$. By Lemma 7.15, $\mathcal{C}^{[[b_\Phi]]}(k) \subset \mathcal{N}^{[b_\Phi]}(k)$ and $\mathcal{C}^{[[b_\Phi,h]]}(k) \subset \mathcal{N}^{[b_\Phi,h]}(k)$ are closed. Then $\mathcal{C}^{[[b_\Phi]]}(\sigma)(k)$ is closed in $\mathcal{N}^{[b_\Phi]}(\sigma)(k)$ and has to be the closure of $\mathcal{C}^{[[b_\Phi]]}(k)$ in $\mathcal{N}^{[b_\Phi]}(\sigma)(k)$. Applying Lemma 7.15 again, the central leaf in $\mathscr{S}_{K_\Phi}(\sigma)(k)$ associated with $[[b_\Phi]]$ is closed in the Newton stratum and contains $\mathcal{C}^{[[b_\Phi]]}(k)$; therefore, it must contain $\mathcal{C}^{[[b_\Phi]]}(\sigma)(k)$. In particular, the preimages of $\mathcal{C}^{[[b_\Phi,h]]}(k)$ in $W(k)$ and in $W^0(k)$ have the same image $[[b_\Phi]] \in C(\mathcal{P}^c_\Phi, \{\mu^{c,-1}_\Phi\})$. Therefore, $(\mathcal{C}^{[[b]]})^\Sigma(k)$ is contained in $(\Upsilon^\Sigma_K(k))^{-1}([[b]])$. Then we apply the last paragraph of the proof of Proposition 7.13. ☐

Lemma 7.15. *Let S be a perfect scheme and let $\mathcal{P}$ be a quasi-parahoric group scheme. Given $p : S \to \mathrm{Sht}^W_{\mathcal{P}} \to \mathrm{Isoc}_P$, we define (the set of k-points of) central leaves to be the fibers of $S(k) \to \mathrm{Sht}^W_{\mathcal{P}}(k)$, and define Newton strata to be the fibers of $S \to B(P)$. Then Newton strata are locally closed in S, and central leaves are closed in the (set of k-points of) Newton strata.*

Proof. When $\mathcal{P}$ is parahoric, this follows from [HK25, Prop. 2.15]. Note that in [HK25, §2], it is not necessary to assume that the generic fiber of $\mathcal{P}$ is reductive; the setup applies to any flat affine group scheme of finite type over the base with connected fibers. The references [ARH19], [ARH21], and [IKY24, Thm. A.14] also work in this general setting.

When $\mathcal{P}$ is quasi-parahoric, we instead apply the arguments in the proof of Lemma 7.5, using [DvHKZ26, Prop. 2.3.1] with $\pi_0(\mathcal{P}^c)^\phi$ in place of $\pi_0(\mathcal{G}^c)^\phi$. ☐

7.2. KR strata and EKOR strata.

7.2.1. *Algebraicity.* Let us recall the setting in [Gle25, §3]. Given a morphism $f\colon S \to T$ of affine perfect schemes, f is a universally subtrusive cover if and only if the induced morphism $f^\diamond\colon S^\diamond \to T^\diamond$ is a v-cover.

Lemma 7.16. *Let S be a perfect scheme, and let H be the perfection of an affine group scheme of finite type. Then $H^1_{uv}(S,H) = H^1_v(S^\diamond, H^\diamond)$, where H^1_{uv} is defined under the universally subtrusive topology.*

Proof. Since $\diamond$ is fully faithful on the category of perfect schemes and preserves surjections, and by definition preserves fiber products, we have a morphism $H^1_{uv}(S,H) \to H^1_v(S^\diamond, H^\diamond)$. On the other hand, given an affinoid perfectoid space $U = \mathrm{Spa}(A, A^+) \to S^\diamond$ that trivializes the given $H^\diamond$-torsor $\mathcal{F}$ on $S^\diamond$, U_{red} is represented by $\mathrm{Spec}\, A_{\mathrm{red}}$ ($A_{\mathrm{red}} = (A/A \cdot A^{\circ\circ})^{\mathrm{perf}}$, see [Gle25, Prop. 3.18]), and $\mathrm{Spec}\, A_{\mathrm{red}} \to S = \mathrm{Spec}\, R$ is surjective (indeed, we have the surjective specialization map $\mathrm{sp}_U : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A^+_{\mathrm{red}}$). We claim that $\mathcal{F}_{\mathrm{red}}$ is represented by a perfect scheme. Following this, since the reduction functor preserves finite limits and $\diamond$ is fully faithful, taking reduction gives a section $H^1_v(S^\diamond, H^\diamond) \to H^1_{uv}(S,H)$. Moreover, by adjointness, $(\mathcal{F}_{\mathrm{red}})^\diamond \to \mathcal{F}$ is a morphism of $H^\diamond$-torsors, which is automatically an isomorphism; then $H^1_v(S^\diamond, H^\diamond) \cong H^1_{uv}(S,H)$.

Let H_0 be a linear algebraic group such that $H = H_0^{\mathrm{perf}}$. We claim that a H_0-torsor on S is the same as a H-torsor on S: recall that a H-torsor over S can be viewed as a trivialization $U \to S$ with a section $H(U \times_S U)$ that satisfies cocycle conditions. Since for any perfect scheme T, $H(T) = H_0(T)$, we have $H^1_{uv}(S, H_0) = H^1_{uv}(S,H)$.

Now we work in the topos SchPerf. Assume $H_0 = \mathrm{GL}_n$. A GL_n-torsor $\mathcal{P}$ on S can be viewed as a vector bundle on S via $\mathcal{P} \times^{\mathrm{GL}_n} \mathcal{O}^n_S$. By [BS17, Thm. 4.1], vector bundles over S form a v-stack, thus $\mathcal{P} \times^{\mathrm{GL}_n} \mathcal{O}^n_S$ is representable by the full-faithfulness of $\diamond$. By taking the framing, $\mathcal{P} = \underline{\mathrm{Isom}}_S(\mathcal{O}_S, \mathcal{P} \times^{\mathrm{GL}_n} \mathcal{O}^n_S)$ is representable. In general, take a closed embedding $H_0 \to \mathrm{GL}_n$ such that the quotient is quasi-affine (see [PR08, Prop. 1.3]), then $\mathrm{GL}_n^{\mathrm{perf}}/H_0^{\mathrm{perf}}$ is a perfect quasi-affine scheme. Let $G = \mathrm{GL}_n^{\mathrm{perf}}$, $H = H_0^{\mathrm{perf}}$. Consider the push-out torsor $\mathcal{Q} = \mathcal{P} \times^H G$, $\mathcal{Q}$ is representable, in particular, $\mathcal{Q} \to S$ is fpqc. Consider the quotient sheaf (under fpqc topology) $\mathcal{Q}/H = \mathcal{Q} \times^G G/H = \mathcal{P} \times^H G/H$, it is trivialized over the fpqc cover $\mathcal{Q} \to S$, thus $\mathcal{Q}/H$ itself is a perfect quasi-affine scheme by the fpqc descent. Since the v-topology on the category of perfect qcqs schemes is subcanonical (see [BS17, Remark 4.2]), then $\mathcal{Q}/H = \mathcal{P} \times^H G/H$ (viewed as a representable v-sheaf) has a section over S, thus $\mathcal{P}$ is representable. $\square$

7.2.2. *Kottwitz-Rapoport strata.* Recall that in [PR24, §4.9], given a quasi-parahoric group scheme $\mathcal{G}$, the authors constructed a v-sheaf theoretical local model diagram:

$$\mathrm{Sht}_{\mathcal{G},\mu} \to [\mathcal{G}^{\lozenge}\backslash \mathbb{M}^{v}_{\mathcal{G},\mu}], \tag{7.7}$$

where $\mathcal{G}^{\lozenge}(S)$ ($S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$) consists of pairs $(S^\sharp, g)$, $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S and $g \in \mathcal{G}(R^\sharp)$. Such constructions are functorial.

On the other hand, we have a local model diagram

$$\mathrm{Sht}^{W}_{\mathcal{G},\mu} \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}] \tag{7.8}$$

constructed as follows: let $\mathrm{Spec}\, R \in \mathrm{PCAlg}^{\mathrm{op}}$, $\mathrm{Sht}^{W,\square}_{\mathcal{G},\mu}(R)$ classifies tuples $((\mathscr{P}, \phi_{\mathscr{P}}), \alpha)$, where $(\mathscr{P}, \phi_{\mathscr{P}}) \in \mathrm{Sht}^{W}_{\mathcal{G},\mu}(R)$ and $\alpha : \mathscr{P}_0 \xrightarrow{\sim} \phi^*(\mathscr{P})$ is a trivialization of the $\mathcal{G}$-torsor $\phi^*(\mathscr{P})$ over $W(R)$. Then $\mathrm{Sht}^{W,\square}_{\mathcal{G},\mu}(R) \to M^{\mathrm{loc}}_{\mathcal{G},\mu}(R)$ that maps $((\mathscr{P}, \phi_{\mathscr{P}}), \alpha)$ to $(\mathscr{P}, \phi_{\mathscr{P}} \circ \alpha)$ is $L^+\mathcal{G}$-equivariant. Note that the $L^+\mathcal{G}$-action on $M^{\mathrm{loc}}_{\mathcal{G},\mu}$ factors through $\mathcal{G}_0$, and the trivialization α is uniquely determined by its reduction over R by the smoothness of $\mathcal{G}$, then we get (7.8).

Remark 7.17. *Let us also compare (7.8) with the one constructed in* [XZ17, §7.2.3] *(cf.* [SYZ21, §4.2.2]*). The local Hecke stack* $\mathrm{Hk}^{W}_{\mathcal{G},\mu}(R)$ *(resp.* $M^{\mathrm{loc}}_{\mathcal{G},\mu}(R) \subset \mathrm{Gr}^{W}_{\mathcal{G}}(R)$*) classifies the modifications* $\gamma : \overleftarrow{\mathscr{P}} \dashrightarrow \overrightarrow{\mathscr{P}}$ *(resp.* $\mathscr{P}_0 \dashrightarrow \mathscr{P}$*) of* $\mathcal{G}$*-torsors over* $W(R)$ *of type* μ*. By choosing a trivialization* $\alpha : \mathcal{P}_0 \xrightarrow{\sim} \overleftarrow{\mathscr{P}}$*, we have* $\mathrm{Hk}^{W}_{\mathcal{G},\mu} = [L^+\mathcal{G}\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$*. Also recall the truncated* $\mathrm{Hk}^{W,(1)}_{\mathcal{G},\mu} = [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$*. We have a morphism* $\mathrm{Sht}^{W}_{\mathcal{G},\mu} \to \mathrm{Hk}^{W}_{\mathcal{G},\mu}$ *that maps* $(\mathscr{P}, \phi_{\mathscr{P}})$ *to* $(\overleftarrow{\mathscr{P}} = \phi^*\mathscr{P}, \overrightarrow{\mathscr{P}} = \mathscr{P}, \gamma = \phi_{\mathscr{P}})$*. Compose with* $\mathrm{Hk}^{W}_{\mathcal{G},\mu} \to \mathrm{Hk}^{W,(1)}_{\mathcal{G},\mu}$*, we get (7.8).*

Lemma 7.18. *Let* $\mathcal{G}_0$ *be the special fiber of* $\mathcal{G}$*. We have a projection* $[\mathcal{G}^{\lozenge}\backslash \mathbb{M}^{v}_{\mathcal{G},\mu}]_{\mathrm{red}} \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$*. In particular, from (7.7), we further have* $\mathrm{Sht}^{W}_{\mathcal{G},\mu} \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$ *which coincides with (7.8).*

Proof. Take the reduction of (7.7). Let $S \in \mathrm{PCAlg}^{\mathrm{op}}$ be an affine perfect scheme,

$$[\mathcal{G}^{\lozenge}\backslash \mathbb{M}^{v}_{\mathcal{G},\mu}]_{\mathrm{red}}(S) = \mathrm{Hom}(S^{\diamond}, [\mathcal{G}^{\lozenge}\backslash \mathbb{M}^{v}_{\mathcal{G},\mu}])$$

gives $S^{\diamond} \xleftarrow{p} \widetilde{S} \xrightarrow{q} \mathbb{M}^{v}_{\mathcal{G},\mu}$, where p is a $\mathcal{G}^{\lozenge}$-torsor, and q is $\mathcal{G}^{\lozenge}$-equivariant. Take reduction again, we have $(S^{\diamond})_{\mathrm{red}} \xleftarrow{\bar{p}} \widetilde{S}_{\mathrm{red}} \xrightarrow{q} M^{\mathrm{loc}}_{\mathcal{G},\mu}$, where $\bar{p}$ is a $(\mathcal{G}_0^{\diamond})_{\mathrm{red}}$-torsor, and q is $(\mathcal{G}_0^{\diamond})_{\mathrm{red}}$-equivariant: given a perfect algebra R, by [Gle25, Theorem 2],

$$(\mathcal{G}^{\lozenge})_{\mathrm{red}}(\mathrm{Spec}\, R) = \mathcal{G}^{\lozenge}(\mathrm{Spd}\, R) = \mathcal{G}(R) = \mathcal{G}_0(R) = (\mathcal{G}_0^{\diamond})_{\mathrm{red}}(\mathrm{Spec}\, R).$$

Since $S \cong (S^{\diamond})_{\mathrm{red}}$, $(\mathcal{G}_0^{\diamond})_{\mathrm{red}}$ is represented by $\mathcal{G}_0^{\mathrm{perf}}$, Lemma 7.16 shows that $\widetilde{S}_{\mathrm{red}}$ is represented by a perfect scheme $\widetilde{S}_0$ and $\widetilde{S}_0 \to S$ is a $\mathcal{G}_0^{\mathrm{perf}}$-torsor. On the other hand, since $\diamond$ is fully faithful and $M^{\mathrm{loc}}_{\mathcal{G},\mu}$ is represented by a perfect scheme (which we still denote by $M^{\mathrm{loc}}_{\mathcal{G},\mu}$), $\widetilde{S}_{\mathrm{red}} \to M^{\mathrm{loc}}_{\mathcal{G},\mu}$ is represented by a $\mathcal{G}_0^{\mathrm{perf}}$-equivariant morphism $\widetilde{S}_0 \to M^{\mathrm{loc}}_{\mathcal{G},\mu}$ between perfect schemes, thus we have a point in $[\mathcal{G}_0^{\mathrm{perf}}\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}](S)$. This gives a morphism $[\mathcal{G}^{\lozenge}\backslash \mathbb{M}^{v}_{\mathcal{G},\mu}]_{\mathrm{red}} \to [\mathcal{G}_0^{\mathrm{perf}}\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}] = [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$. By constructions, these two $\mathrm{Sht}^{W}_{\mathcal{G},\mu} \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}_{\mathcal{G},\mu}]$ coincide. $\square$

We apply this to $\mathscr{S}_K(G, X)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c}$, and use Lemma 7.16 again. We then have a morphism:

$$\mathscr{S}_K(G, X)^{\mathrm{perf}}_{\bar{s}} \to [\mathcal{G}_0^c\backslash M^{\mathrm{loc}}_{\mathcal{G}^c,\mu^c}]. \tag{7.9}$$

When $\mathcal{G}^c$ is parahoric, it is homeomorphic to

$$\begin{aligned}\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c} &:= \breve{K}^c\backslash \breve{K}^c\,\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})\breve{K}^c/\breve{K}^c \\ &= W_{K^c}\backslash W_{K^c}\,\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})W_{K^c}/W_{K^c} \subset W_{K^c}\backslash \widetilde{W}^c/W_{K^c},\end{aligned}$$

where we use the induced Bruhat order (from the affine-Weyl group $\widetilde{W}^c$ of G^c) on $W_{K^c}\backslash\widetilde{W}^c/W_{K^c}$, thus the morphism (7.9) induces a continuous morphism of underlying topological spaces

$$l_K : \mathscr{S}_K(G,X)_{\bar{s}} \to \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c}.$$

When $\mathcal{G}^c$ is quasi-parahoric, since $\mathbb{M}_{\mathcal{G}^{c,\circ},\mu^c} \cong \mathbb{M}_{\mathcal{G}^c,\mu^c}$, we identify

$$M^{\mathrm{loc}}_{\mathcal{G}^c,\mu^c}(k) = \breve{K}^{c,\circ}\,\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})\breve{K}^c/\breve{K}^c.$$

It is equal to $\breve{K}^c\,\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})\breve{K}^c/\breve{K}^c$ since the conjugation of $\breve{K}^c/\breve{K}^{c,\circ} \hookrightarrow \pi_1(G^c)_I$ acts trivially on $\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})$.. Thus

$$|[\mathcal{G}_0^{c,\circ,\mathrm{perf}}\backslash M^{\mathrm{loc}}_{\mathcal{G}^{c,\circ},\mu^c}]| = \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^{c,\circ}} = \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c} = |[\mathcal{G}_0^c\backslash M^{\mathrm{loc}}_{\mathcal{G}^c,\mu^c}]|.$$

We define the fibers of l_K as *Kottwitz-Rapoport strata*. In particular, KR strata are locally closed.

By applying the exact same arguments, given a quasi-parahoric group scheme $\mathcal{P}$ such that $(\mathcal{P},\mu)$ comes from boundary (see Definition 1.21), and by Corollary 1.32, the $L^+\mathcal{P}$-action on $\mathbb{M}^v_{\mathcal{P},\mu}$ factors through $\mathcal{P}^\Diamond$, we have morphisms

$$\mathrm{Sht}_{\mathcal{P},\mu} \to [\mathcal{P}^\Diamond\backslash\mathbb{M}^v_{\mathcal{P},\mu}], \quad \mathrm{Sht}^W_{\mathcal{P},\mu} \to [\mathcal{P}_0\backslash M^{\mathrm{loc}}_{\mathcal{P},\mu}].$$

Proposition 7.19. *KR strata are well positioned. Moreover, let* KR_w *be a KR stratum on* $\mathscr{S}_{K,\bar{s}}$ *with some* $w \in \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c}$. *Then, for each* $\Phi \in \mathcal{CLR}(G,X)$, $(\mathrm{KR}_w)^\natural_{\mathrm{Z}^\star(\Phi)}$ *is either empty or a finite union of KR strata* $\mathrm{KR}_{w_{\Phi,h}}$ *on* $\mathrm{Z}^\star(\Phi)_{\bar{s}} \cong \mathscr{S}_{K_{\Phi,h},\bar{s}}$, *for some collection of* $w_{\Phi,h} \in \mathrm{Adm}_{G^*_{\Phi,h}}(\{\mu^{*,-1}_{\Phi,h}\})_{\breve{K}^*_{\Phi,h}}$. *The relation between* w *and the collection of* $w_{\Phi,h}$ *is given in the proof.*

Proof. KR strata are locally closed and are unions of central leaves. Since central leaves are well positioned by Proposition 7.12, KR strata are well positioned; see [Mao25b, Lem. 2.20]. We describe their boundaries. Using the main diagram (5.20), we have

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^\diamond \arrow[d] & W^{0,\diamond} \arrow[l] \arrow[r] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)^\diamond \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h})^\diamond \arrow[d] \\
{[\mathcal{G}^{c,\Diamond}\backslash\mathbb{M}^v_{\mathcal{G}^c,\mu^{c,-1}}]} & & {[\mathcal{P}^{*,\Diamond}_\Phi\backslash\mathbb{M}^v_{\mathcal{P}^*_\Phi,\mu^{*,-1}_\Phi}]} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r] & {[\mathcal{G}^{*,\Diamond}_{\Phi,h}\backslash\mathbb{M}^v_{\mathcal{G}^*_{\Phi,h},\mu^{*,-1}_{\Phi,h}}]}.
\end{tikzcd}$$

Applying the above arguments, we have

$$\begin{tikzcd}
\mathscr{S}_K(G,X)_{\bar{s}} \arrow[d] & W^0_{\bar{s}} \arrow[l] \arrow[r] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_\Phi}(P_\Phi,D_\Phi)_{\bar{s}} \arrow[r] \arrow[d] & \Delta^\circ_{\Phi,K}\backslash\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h},D_{\Phi,h})_{\bar{s}} \arrow[d] \\
\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c} & & \mathrm{Adm}_{P^*_\Phi}(\{\mu^{*,-1}_\Phi\})_{\breve{K}^*_\Phi} \arrow[ll, "\mathrm{Int}(g_\Phi^{-1})"'] \arrow[r, "=(1.5)"] & \mathrm{Adm}_{G^*_{\Phi,h}}(\{\mu^{*,-1}_{\Phi,h}\})_{\breve{K}^*_{\Phi,h}}.
\end{tikzcd}$$

The arguments in the proof of Proposition 7.7 can be adapted here without much change. ☐

Remark 7.20. *We expect that the boundary of a KR stratum should be a single KR stratum, as proved in [LS18a] for PEL types and in [Mao25b] for Hodge types. In other words, we expect that*

$$\mathrm{Adm}_{G^*_{\Phi,h}}(\{\mu^{*,-1}_{\Phi,h}\})_{\breve{K}^*_{\Phi,h}} \cong \mathrm{Adm}_{L^*_\Phi}(\{\mu^{*,-1}_{\Phi,L}\})_{\breve{K}^*_{\Phi,L}} \hookrightarrow \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c}$$

is an injection.

Next, we work with partial toroidal compactifications of KR strata. Equations (7.4) and (7.9) give

$$\mathscr{S}_K^{\Sigma}(G,X)_{\bar{s}}^{\mathrm{perf}} \to [\mathcal{G}_0^c \backslash M^{\mathrm{loc}}_{\mathcal{G}^c,\mu^c}], \quad l_K^{\Sigma} : \mathscr{S}_K^{\Sigma}(G,X)_{\bar{s}} \to \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c}. \tag{7.10}$$

Proposition 7.21. *Let* $w \in \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c}$*, and let* $\mathrm{KR}_w \subset \mathscr{S}_K(G,X)_{\bar{s}}$ *be the KR stratum. Then its partial toroidal compactification* KR_w^{Σ} *is the fiber* $l_K^{\Sigma,-1}(w)$*.*

Proof. Similar to the proof of Proposition 7.13. ☐

7.2.3. *Relation with schematic local model diagram.* In [AGLR22] and [GL24], it was proved that $\mathbb{M}^v_{\mathcal{G},\mu}$ is represented by a normal scheme $\mathcal{M}_{\mathcal{G},\mu}$ that is flat and of finite type over $\mathcal{O}_E$. In [PR24, §4.9.1], it is conjectured that there exists a schematic local model diagram; that is to say, there exists a smooth morphism

$$\pi_{\mathrm{dR},\mathcal{G}^c} : \mathscr{S}_K(G,X) \to [\mathcal{G}^c \backslash \mathcal{M}_{\mathcal{G}^c,\mu^c}],$$

such that we have a (2-)commutative diagram

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^{\Diamond/} \arrow[r] \arrow[d, "\pi^{\Diamond/}_{\mathrm{dR},\mathcal{G}^c}"] & \mathrm{Sht}_{\mathcal{G}^c,\mu^c} \arrow[d] \\
{[\mathcal{G}^{c,\Diamond/} \backslash \mathbb{M}^v_{\mathcal{G}^c,\mu^c}]} \arrow[r] & {[\mathcal{G}^{c,\Diamond} \backslash \mathbb{M}^v_{\mathcal{G}^c,\mu^c}]}.
\end{tikzcd} \tag{7.11}$$

Here $\mathcal{M}^{\Diamond/}_{\mathcal{G}^c,\mu^c} = \mathcal{M}^{\Diamond}_{\mathcal{G}^c,\mu^c} = \mathbb{M}^v_{\mathcal{G}^c,\mu^c}$ by properness. In most cases of Hodge type, the Kisin-Pappas-Zhou integral models have schematic local model diagrams; see [KPZ24, Thm. 7.1.3] and [DvHKZ26, Appendix A]. In these cases, by the (perfect) smoothness of the morphism, we have a closure relation

$$\overline{\mathrm{KR}_w} = \bigsqcup_{w' \leq w, w' \in \mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}}} \mathrm{KR}_{w'}, \tag{7.12}$$

Here the KR strata are defined using the fibers of $\pi_{\mathrm{dR},\mathcal{G}^c}$, and this coincides with the fibers of l_K (7.9), by the commutativity of the above diagram (7.11).

We focus on the abelian-type case. Assume $\mathcal{G}$ is parahoric. In [KPZ24, Thm. 7.2.20 and Rmk. 7.2.22] (with supplements in [DvHKZ26]), the authors showed that, given an abelian-type Shimura datum (G,X), when $p > 2$ (we use (G,X) instead of (G_2,X_2) to keep consistency of the notation in this subsection), there exists an integral model $\mathscr{S}_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ that has a list of good properties (see [KPZ24, Thm. 7.2.20]). Among these properties, we have a smooth morphism

$$\pi_{\mathrm{dR},\mathcal{G}^{\mathrm{ad}}} : \mathscr{S}_K(G,X) \to [\mathcal{G}^{\mathrm{ad},\circ} \backslash \mathcal{M}_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}]. \tag{7.13}$$

Since $\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})_{\breve{K}^c} \cong \mathrm{Adm}_{G^{\mathrm{ad}}}(\{\mu^{\mathrm{ad},-1}\})_{\breve{K}^{\mathrm{ad},\circ}}$ (see [SYZ21, Lem. 5.1.4]), we have a stratification (7.12), here the KR strata are defined using the fibers of $\pi_{\mathrm{dR},\mathcal{G}^{\mathrm{ad}}}$.

Proposition 7.22. *When* $\mathcal{G}$ *is parahoric and* $p > 2$*, we have a (2-)commutative diagram*

$$\begin{tikzcd}
\mathscr{S}_K(G,X)^{\Diamond/} \arrow[r] \arrow[d, "\pi^{\Diamond/}_{\mathrm{dR},\mathcal{G}^{\mathrm{ad},\circ}}"] & \mathrm{Sht}_{\mathcal{G}^c,\mu^c} \arrow[d] \\
{[\mathcal{G}^{\mathrm{ad},\circ,\Diamond/} \backslash \mathbb{M}^v_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}]} \arrow[r] & {[\mathcal{G}^{\mathrm{ad},\circ,\Diamond} \backslash \mathbb{M}^v_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}]}.
\end{tikzcd}$$

Proof. When (G,X) is Hodge-type, this is essentially [DvHKZ26, Thm. A.3.3, Prop. 4.3.3]. We generalize this result to the abelian-type case. We argue as follows: we first show the proposition when $G = G^{\mathrm{ad}}$, and then show it for a general abelian-type Shimura datum.

When $G = G^{\mathrm{ad}}$, by the construction in [KPZ24], there is a Hodge-type Shimura datum (G', X') and a parahoric group scheme $\mathcal{G}'$ lifting $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \mathcal{G}^{\mathrm{ad},\circ})$ that satisfy Conditions (A)–(E) in §7.1 of *loc. cit.*. Consider the diagram:

$$\begin{tikzcd}
\mathscr{S}_{K'}(G', X')^{\lozenge/} \arrow[rr] \arrow[dd, "\pi^{\lozenge/}_{\mathrm{dR},\mathcal{G}^{\mathrm{ad},\circ}}(G',X')"] \arrow[dr] & & \mathrm{Sht}_{\mathcal{G}',\mu'} \arrow[dd] \arrow[dr] & \\
& \mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\lozenge/} \arrow[rr] \arrow[dl, "\pi^{\lozenge/}_{\mathrm{dR},\mathcal{G}^{\mathrm{ad},\circ}}(G^{\mathrm{ad}},X^{\mathrm{ad}})"] & & \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}} \arrow[dl] \\
{[\mathcal{G}^{\mathrm{ad},\circ,\lozenge/}\backslash\mathbb{M}^v_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}]} \arrow[rr] & & {[\mathcal{G}^{\mathrm{ad},\circ,\lozenge}\backslash\mathbb{M}^v_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}].} &
\end{tikzcd}$$

The left triangle is commutative by the construction of the $\mathcal{G}^{\mathrm{ad},\circ}$ local model diagram from the Hodge-type case in [KPZ24]. The right triangle is canonically commutative. Since we can cover $\mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ by a disjoint union of $\mathscr{S}_{K',\alpha}(G', X')$ satisfying (A)-(E) (where $K'^{,\alpha}$ is $K'^{,\alpha}_p K'^{,\alpha,p}$ for $K'^{,\alpha}_p$ the conjugation of $K'_p = \mathcal{G}'(\mathbb{Z}_p)$ by an element $\alpha \in G'(\mathbb{Q}_p)$ and $K'^{,\alpha,p}$ neat open compact), it suffices to show the commutativity of the bottom square by composition with $\mathscr{S}^{\lozenge/}_{K'} \to \mathscr{S}^{\lozenge/}_{K^{\mathrm{ad}}}$. Note that $\mathscr{S}_{K'}(G', X')^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}',\mu'} \to \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}} = \mathscr{S}_{K'}(G', X')^{\lozenge/} \to \mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}$ by functoriality of Kisin-Pappas-Zhou integral models (see [DY25, Thm. B and Cor. C]). Then the desired commutativity for adjoint Shimura data follows from the Hodge-type case and diagram-chasing.

For a general (G, X), the assertion follows from the last paragraph and the commutativity of

$$\begin{tikzcd}
\mathscr{S}_K(G, X)^{\lozenge/} \arrow[r] \arrow[d, "\pi^{\mathrm{ad},\lozenge/}"] & \mathrm{Sht}_{\mathcal{G}^c,\mu^c} \arrow[d] \\
\mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\lozenge/} \arrow[r] & \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu^{\mathrm{ad}}}
\end{tikzcd}$$

by [DY25, Thm. B and Cor. C] again. Note that, since the local model diagram of an abelian-type integral model $\mathscr{S}_K(G, X)$ in [KPZ24] is constructed by first pushing out the $\mathcal{G}'^{,\circ}$-local model diagram of $\mathscr{S}_{K'}(G', X')$ to a $\mathcal{G}^{\mathrm{ad},\circ}$-local model diagram and then passing to $\mathscr{S}_K(G, X)$ as in [KP18, Cor. 4.6.18], we still have $\pi^{\lozenge/}_{\mathrm{dR},\mathcal{G}^{\mathrm{ad},\circ}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \circ \pi^{\mathrm{ad},\lozenge/} = \pi^{\lozenge/}_{\mathrm{dR},\mathcal{G}^{\mathrm{ad},\circ}}(G, X)$. The proposition is now proved. ☐

Corollary 7.23. *Such constructed KR strata (using $\pi_{\mathrm{dR},\mathcal{G}^{\mathrm{ad}}}$) coincide with the ones defined in the last subsection (using l_K).*

Corollary 7.24. *Under the setting of Proposition 7.22, we have a closure relation:*

$$\overline{\mathrm{KR}^{\Sigma}_{w}} = \bigsqcup_{w' \leq w, w' \in \mathrm{Adm}_{\mathcal{G}^c}(\{\mu^{c,-1}\})_{\breve{K}}} \mathrm{KR}^{\Sigma}_{w'}.$$

Proof. This follows from (7.12) and the compatibility in Corollary 7.23 (see [Mao25b, Prop. 7.1]). ☐

Remark 7.25. *In* [DY25, Thm. A]*, the authors constructed a shtuka map on Kisin-Pappas-Zhou integral models when $p > 2$ with parahoric level structures; the construction of the integral model and the shtuka map $\mathscr{S}_K(G, X) \to \mathrm{Sht}_{\mathcal{G}^c,\mu^c}$ in this paper coincides with* loc. cit. *by the uniqueness of canonical integral models (see Theorem 6.26 and* [DY25, Thm. B]*) and by the unique extension of shtukas from generic fibers to integral models (*[PR24, Cor. 2.7.10]*). Proposition 7.22 relies on the work* [KPZ24] *on the existence of local model diagrams when $p > 2$. When $p = 2$, see* [Yan25] *for recent progress.*

7.2.4. *Ekedahl-Kottwitz-Oort-Rapoport strata.* It is difficult to study EKOR strata only using shtukas; nevertheless, let us mention some easily deduced properties. By the constructions in [SYZ21, §4.1], for parahoric $\mathcal{G}$, from $\mathscr{S}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c}$, one can define

$$\mathscr{S}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to (\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c})^{\mathrm{loc}(m,1)}, \quad \nu_K : \mathscr{S}_K(G,X)_{\bar{s}} \to {}^{\breve{K}^c}\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\}),$$

where $(\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c})^{\mathrm{loc}(m,1)}$ is the stack of $(m,1)$-truncated shtukas. It is an algebraic stack, with underlying topological space homeomorphic to ${}^{\breve{K}^c}\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})$. We define the preimages of ν_K as *EKOR strata*. EKOR strata are locally closed.

Similarly, we can consider the partial toroidal compactifications of EKOR strata. (7.4) gives

$$\mathscr{S}^{\Sigma}_K(G,X)^{\mathrm{perf}}_{\bar{s}} \to (\mathrm{Sht}^W_{\mathcal{G}^c,\mu^c})^{\mathrm{loc}(m,1)}, \quad \nu^{\Sigma}_K : \mathscr{S}^{\Sigma}_K(G,X)_{\bar{s}} \to {}^{\breve{K}^c}\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\}).$$

In the abelian-type case, when $\mathcal{G}$ is quasi-parahoric, we further define the EKOR strata on $\mathscr{S}_{K,\bar{s}}$ (resp. $\mathscr{S}^{\Sigma}_{K,\bar{s}}$) as the images of EKOR strata on $\mathscr{S}_{K^\circ,\bar{s}}$ (resp. $\mathscr{S}^{\Sigma}_{K^\circ,\bar{s}}$) under the finite étale surjection $\mathscr{S}_{K^\circ,\bar{s}} \to \mathscr{S}_{K,\bar{s}}$ (resp. $\mathscr{S}^{\Sigma}_{K^\circ,\bar{s}} \to \mathscr{S}^{\Sigma}_{K,\bar{s}}$; the étaleness can be easily seen using the étaleness of $\mathscr{S}_{K^\circ_\Phi} \to \mathscr{S}_{K_\Phi}$ for any $\Phi \in \mathcal{CLR}(G,X)$).

Proposition 7.26. *EKOR strata are well positioned. Let* $w \in {}^{\breve{K}^c}\mathrm{Adm}_{G^c}(\{\mu^{c,-1}\})$, $\mathrm{EKOR}_w \subset \mathscr{S}_K(G,X)_{\bar{s}}$ *be the EKOR strata; then its partial toroidal compactification* EKOR^{Σ}_w *is the fiber* $\nu^{\Sigma,-1}_K(w)$.

Proof. EKOR strata are locally closed and are unions of central leaves. Since central leaves are well positioned by Proposition 7.12, EKOR strata are well positioned (see [Mao25b, Lem. 2.20]). Write $\mathrm{EKOR}_w = \coprod_{i\in I} \mathcal{C}^{[[b_i]]}$ as a union of central leaves; then $\mathrm{EKOR}^{\Sigma}_w = \coprod_{i\in I}(\mathcal{C}^{[[b_i]]})^{\Sigma}$ by [Mao25b, Lem. 2.20]. By Proposition 7.14, $(\mathcal{C}^{[[b_i]]})^{\Sigma} \subset \nu^{\Sigma,-1}_K(w)$; then $\mathrm{EKOR}^{\Sigma}_w \subset \nu^{\Sigma,-1}_K(w)$. Since $\mathscr{S}_K(G,X)_{\bar{s}} = \coprod_w \mathrm{EKOR}_w$, $\mathscr{S}^{\Sigma}_K(G,X)_{\bar{s}} = \coprod_w \mathrm{EKOR}^{\Sigma}_w$ by [Mao25b, Lem. 2.20]. This forces $\mathrm{EKOR}^{\Sigma}_w = \nu^{\Sigma,-1}_K(w)$. □

Remark 7.27. *Finally, let us briefly discuss the Ekedahl–Oort strata. Assume that* $\mathcal{G}$ *is hyperspecial. Then, for any* $[\Phi] \in \mathrm{Cusp}_K(G,X)$, $\mathcal{G}_{\Phi,h}$ *is hyperspecial. In* [XZ17, §5.3], *there is a natural perfectly smooth morphism*

$$(\mathrm{Sht}^W_{\mathcal{G},\mu})^{\mathrm{loc}(m,1)} \to G\text{-}\mathrm{Zip}^{perf}_{\mu^{-1}},$$

where $G\text{-}\mathrm{Zip}^{perf}$ *is the perfection of* $G\text{-}\mathrm{Zip}$. *By composing with* $(\mathscr{S}_{K,\bar{s}})^{perf} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c}$ *and* $(\mathscr{S}^{\Sigma}_{K,\bar{s}})^{perf} \to \mathrm{Sht}^W_{\mathcal{G}^c,\mu^c}$, *we obtain*

$$\epsilon_K : (\mathscr{S}_{K,\bar{s}})^{perf} \to G^c\text{-}\mathrm{Zip}^{perf}_{\mu^{c,-1}}, \quad \epsilon^{\Sigma}_K : (\mathscr{S}^{\Sigma}_{K,\bar{s}})^{perf} \to G^c\text{-}\mathrm{Zip}^{perf}_{\mu^{c,-1}}.$$

We can define the EO strata as the fibers of ϵ_K, *endowed with induced reduced subscheme structures, and similarly show that the EO strata are well positioned.*

Thus, we obtain a similar diagram for ϵ_K *and* ϵ^{Σ}_K *as those in Proposition 7.19, so that we deduce that, when* Σ *is smooth,* ϵ_K *is perfectly smooth if and only if* ϵ^{Σ}_K *is so (cf.* [Wu25, Cor. 4.51]*).*

Moreover, the partial toroidal compactification of the EO strata coincides with the EO strata defined using ϵ^{Σ}_K. *Also, when* $\mathcal{G}$ *is hyperspecial, EKOR strata are the same as EO strata; thus, we omit the proof for EO strata here.*

Remark 7.28. *For the KR strata, we expect the existence of* ***schematic*** *local model diagram. For EO strata, one expects that these morphisms arise from a prismatic or syntomic approach, where the coherent and infinitesimal structures can be directly seen; that is, the morphisms* ϵ_K *and* ϵ^{Σ}_K *should respectively be the perfections of smooth morphisms*

$$\mathscr{S}_{K,s} \to G^c\text{-}\mathrm{Zip}_{\mu^{c,-1}}, \quad \mathscr{S}^{\Sigma}_{K,s} \to G^c\text{-}\mathrm{Zip}_{\mu^{c,-1}}.$$

For the interior morphism ϵ_K, the most general result currently available is given by [MY26]*; let us also mention the previous works* [Oor01], [VW13], [PWZ15], [Zha18], [IKY23], *etc. Indeed, when (G, X) is of abelian type or when p is large enough, Madapusi and Youcis constructed a formally étale map of p-adic formal stacks over $\operatorname{Spf}\mathcal{O}_E$: $\widehat{\mathscr{S}_K} \to \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu^c}$, where $\mathrm{BT}_\infty^{\mathcal{G}^c,-\mu^c}$ serves as the stack of 'p-divisible groups with $\mathcal{G}^c$-structures' without actually working with p-divisible groups. The compositions $\widehat{\mathscr{S}_K} \to \mathrm{BT}_n^{\mathcal{G}^c,-\mu^c}$ are smooth and surjective, for all $n > 0$. There is a smooth surjection $\mathrm{BT}_1^{\mathcal{G}^c,-\mu^c} \otimes k_E \to G^c\text{-Zip}_{\mu^{c,-1}}$. The composition gives the deperfection of ϵ_K.*

APPENDIX A. SOME CATEGORY THEORY

This appendix serves as a complement to the categorical language employed throughout the paper, offering a direct reference applicable to the context considered herein.

A.1. **Morphisms between presheaves in categories.** Let $\mathcal{C}$ be a (1-)category. Let $\mathsf{Categories}$ (resp. $\mathsf{Groupoids}$) be the 2-category (resp. (2, 1)-category) of categories (resp. groupoids). Let F_1 and F_2 be two (2-)functors (or presheaves) $F_1, F_2 : \mathcal{C}^{\mathrm{op}} \to \mathsf{Categories}$.

As in [Sta25, Ex. 02XV], there are fibered categories $p_1 : \mathcal{X}_1 \to \mathcal{C}$ and $p_2 : \mathcal{X}_2 \to \mathcal{C}$ corresponding to F_1 and F_2, respectively. More precisely, a category $\mathcal{X}$ with a functor p to $\mathcal{C}$ is called a fibered category if, for any $x \in \mathcal{X}$ with $f : V \to p(x) \in \operatorname{Mor}_{\mathcal{C}}(V, p(x))$, there exists a strongly Cartesian lift $y \to x$ over f (cf. [Sta25, Def. 02XM]).

A *(1-)morphism* $\mathscr{F} : (\mathcal{X}_1, p_1) \to (\mathcal{X}_2, p_2)$ is a 1-morphism in the 2-category of fibered categories over $\mathcal{C}$. That is, $\mathscr{F}$ is a functor from $\mathcal{X}_1$ to $\mathcal{X}_2$ such that $p_2 \circ \mathscr{F} = p_1$, and $\mathscr{F}$ sends a strongly Cartesian morphism to a strongly Cartesian morphism (see [Sta25, Def. 02XP]).

On the other hand, we define 1-morphisms between presheaves in categories.

Definition A.1. *Define a (1-)morphism (or a natural transformation) $\mathbf{F} : F_1 \to F_2$ to be the collection $\{\mathbf{F}(U); \phi(U_1, U_2, f)\}_{U,U_1,U_2 \in \operatorname{Ob}\mathcal{C}, f \in \operatorname{Mor}_{\mathcal{C}}(U_1,U_2)}$ as follows:*

(1) For any $U \in \operatorname{Ob}\mathcal{C}$, $\mathbf{F}(U) : F_1(U) \to F_2(U)$ is a 1-morphism (i.e., a functor) between categories.

(2) For $f \in \operatorname{Mor}_{\mathcal{C}}(U_1, U_2)$ with $U_1, U_2 \in \operatorname{Ob}\mathcal{C}$,

$$\phi(U_1, U_2, f) : \mathbf{F}(U_1)F_1(f)(F_1(U_2)) \to F_2(f)\mathbf{F}(U_2)(F_1(U_2))$$

is a natural transformation between functors $\mathbf{F}(U_1)F_1(f)$ and $F_2(f)\mathbf{F}(U_2)$ mapping from $F_1(U_2)$ to $F_2(U_1)$ such that,

  (a) $\phi(U, U, \mathrm{id}_U)(x) = \mathrm{id}_{\mathbf{F}(U)x}$ for any $U \in \operatorname{Ob}\mathcal{C}$ and $x \in F_1(U)$.

  (b) For any morphisms in $\mathcal{C}$, $f : U_1 \to U_2$ and $g : U_2 \to U_3$, and any $x_3 \in \operatorname{Ob}F_1(U_3)$, we have $F_2(f)(\phi(U_2, U_3, g)) \circ \phi(U_1, U_2, f)(F_1(g)x_3) = \phi(U_1, U_3, g \circ f)(x_3)$.

  (c) For any pair (U_1, U_2) and $f : U_1 \to U_2$, the diagram

$$\begin{tikzcd}
F_1(U_2) \arrow[r, "F_1(f)"] \arrow[d, "\mathbf{F}(U_2)"] & F_1(U_1) \arrow[d, "\mathbf{F}(U_1)"] \\
F_2(U_2) \arrow[r, "F_2(f)"] & F_2(U_1)
\end{tikzcd}$$

  is commutative up to composing $\phi(U_1, U_2, f)$.

In fact, the definitions of 1-morphisms in the two contexts correspond to each other.

Lemma A.2. *$\mathscr{F}$ determines and is determined by a morphism $\mathbf{F}$ from F_1 to F_2.*

Proof. Recall that the fibered category $p_1 : \mathcal{X}_1 \to \mathcal{C}$ is defined as follows: the objects are (U, x) such that $U \in \operatorname{Ob}\mathcal{C}$ and $x \in \operatorname{Ob}F_1(U)$; the morphisms are $\operatorname{Mor}_{\mathcal{X}_1}((U_1, x_1), (U_2, x_2)) = \{(f, \phi) | f \in \operatorname{Mor}_{\mathcal{C}}(U_1, U_2), \phi \in \operatorname{Mor}_{F_1(U_1)}(x_1, F_1(f)x_2)\}$. The composition is defined as $(g, \psi) \circ (f, \phi) = (g \circ f, F_1(f)(\psi) \circ \phi)$.

The 1-morphism $\mathscr{F}$ maps (U, x) to $(V, y) \in \operatorname{Ob} \mathcal{X}_2$. Since $p_2 \circ \mathscr{F} = p_1$, we have that $V = U$ and $y \in \operatorname{Ob} F_2(U)$. This determines an assignment $\mathscr{F}(U)$ from $\operatorname{Ob} F_1(U)$ to $\operatorname{Ob} F_2(U)$.

Hence, the functor $\mathscr{F}$ maps $\operatorname{Mor}_{\mathcal{X}_1}((U_1, x_1), (U_2, x_2))$ to $\operatorname{Mor}_{\mathcal{X}_2}((U_1, \mathscr{F}(U_1)x_1), (U_2, \mathscr{F}(U_2)x_2))$. More precisely, a pair (f, ϕ) as above is sent to $(\mathscr{F}(f), \mathscr{F}(\phi))$. But $\mathscr{F}(f) = f$ since $p_2 \circ \mathscr{F} = p_1$. So $\mathscr{F}(\phi) : \mathscr{F}(U_1)x_1 \to F_2(f)(\mathscr{F}(U_2)x_2)$ is a uniquely determined morphism in $F_2(U_1)$ induced by $\mathscr{F}$.

Now let $(U_1, x_1) = (U_1, F_1(f)x_2)$ and let $\phi = \mathrm{id}$. Then $\mathscr{F}(\phi) =: \phi_{\mathscr{F}}(U_1, U_2, f)(x_2)$ is a functor $\mathscr{F}(\phi) : \mathscr{F}(U_1)(F_1(f)x_2) \to F_2(f)(\mathscr{F}(U_2)x_2)$. Moreover, fix any $(t : x_2 \to x_2') \in \operatorname{Mor}_{F_1(U_2)}(x_2, x_2')$ and consider the commutative diagram

$$\begin{tikzcd}
(U_1, F_1(f)x_2) \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "F_1(f)(t)"] & (U_2, x_2) \arrow[d, "t"] \\
(U_1, F_1(f)x_2') \arrow[r, "{(f,\mathrm{id})}"] & (U_2, x_2').
\end{tikzcd}$$

The diagram above is commutative after applying $\mathscr{F}$ as $\mathscr{F}$ is a functor, which implies the commutativity of

$$\begin{tikzcd}
\mathscr{F}(U_1)F_1(f)x_2 \arrow[r, "{\phi_{\mathscr{F}}(U_1,U_2,f)(x_2)}"] \arrow[d, "\mathscr{F}(U_1)F_1(f)t"] & F_2(f)\mathscr{F}(U_2)x_2 \arrow[d, "F_2(f)\mathscr{F}(t)"] \\
\mathscr{F}(U_1)F_1(f)x_2' \arrow[r, "{\phi_{\mathscr{F}}(U_1,U_2,f)(x_2')}"] & F_2(f)\mathscr{F}(U_2)x_2'.
\end{tikzcd}$$

This implies that $\phi_{\mathscr{F}}(U_1, U_2, f)$ is a natural transformation.

Hence, the diagram of functors

$$\begin{tikzcd}
F_1(U_2) \arrow[r, "F_1(f)"] \arrow[d, "\mathscr{F}(U_2)"] & F_1(U_1) \arrow[d, "\mathscr{F}(U_1)"] \\
F_2(U_2) \arrow[r, "F_2(f)"] & F_2(U_1)
\end{tikzcd}$$

is commutative up to uniquely determined 2-morphisms. Then it can be checked that $\{\mathscr{F}(U); \phi_{\mathscr{F}}(U_1, U_2, f)\}$ determines a 1-morphism $\mathbf{F} : F_1 \to F_2$. The second condition of $\mathbf{F}$ listed above follows from the fact that $\mathscr{F}$ is a functor that preserves compositions.

Conversely, suppose that there is a 1-morphism $\mathbf{F} : F_1 \to F_2$ given by the collection $\{\mathbf{F}(U); \phi(U_1, U_2, f)\}$. Then set $\mathscr{F}(U, x_1) = (U, \mathbf{F}(U)x_1)$. Let $(f : U_1 \to U_2, \phi : x_1 \to F_1(f)x_2) \in \operatorname{Mor}_{\mathcal{X}_1}((U_1, x_1), (U_2, x_2))$. Set $\mathscr{F}(f, \phi) = (f, \phi(U_1, U_2, f)(x_2) \circ \mathbf{F}(U_1)(\phi))$.

Then it can be checked that $\mathscr{F}$ is a functor. We only check that it preserves compositions because the other conditions are easier. Suppose that we are given $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$, $x_i \in F_1(U_i)$ for $i = 1, 2, 3$, and $\phi : x_1 \to F_1(f)x_2$ and $\psi : x_2 \to F_1(g)x_3$. We have $\mathscr{F}(g \circ f, F_1(f)(\psi) \circ \phi) =$

$$(g \circ f, \phi(U_1, U_3, g \circ f)(x_3) \circ \mathbf{F}(U_1)(F_1(f)(\psi) \circ \phi)).$$

The last expression is computed as

$$\begin{aligned}
&(g \circ f, F_2(f)(\phi(U_2, U_3, g)) \circ \phi(U_1, U_2, f)(F_1(g)x_3) \circ \mathbf{F}(U_1)(F_1(f)(\psi) \circ \phi)) = \\
&(g \circ f, F_2(f)(\phi(U_2, U_3, g)) \circ \phi(U_1, U_2, f)(F_1(g)x_3) \circ (\mathbf{F}(U_1)(F_1(f)(\psi)) \circ \mathbf{F}(U_1)(\phi))) = \\
&(g \circ f, F_2(f)(\phi(U_2, U_3, g)) \circ F_2(f)\mathbf{F}(U_2)(\psi) \circ \phi(U_1, U_2, f)(x_2) \circ \mathbf{F}(U_1)(\phi)) = \\
&(g \circ f, F_2(f)(\phi(U_2, U_3, g) \circ \mathbf{F}(U_2)(\psi)) \circ (\phi(U_1, U_2, f)(x_2) \circ \mathbf{F}(U_1)(\phi))) = \\
&(g, \phi(U_2, U_3, g) \circ \mathbf{F}(U_2)(\psi)) \circ (f, \phi(U_1, U_2, f)(x_2) \circ \mathbf{F}(U_1)(\phi)) = \mathscr{F}(g, \psi) \circ \mathscr{F}(f, \phi).
\end{aligned}$$

The second line to the third line follows from the fact that $\phi(U_1, U_2, f)$ is a natural transformation, from which the diagram

$$\begin{tikzcd}
\mathbf{F}(U_1)F_1(f)x_2 \arrow[r, "\phi(U_1,U_2,f)(x_2)"] \arrow[d, "\mathbf{F}(U_1)F_1(f)(\psi)"] & F_2(f)\mathbf{F}(U_2)x_2 \arrow[d, "F_2(f)\mathbf{F}(U_2)(\psi)"] \\
\mathbf{F}(U_1)F_1(f)(F_1(g)x_3) \arrow[r, "\phi(U_1,U_2,f)(F_1(g)x_3)"] & F_2(f)\mathbf{F}(U_2)(F_1(g)x_3)
\end{tikzcd}$$

commutes. □

A.2. **2-limits.**

A.2.1. We explain what a 2-limit is, specialized to our situation.

Definition A.3. *Let $p : \mathcal{X} \to \mathcal{C}$ be a fibered category in groupoids constructed from a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Groupoids}$ (see* [Sta25, Def. 003T] *and* [Sta25, Ex. 0049]*). Define*

$$2\text{-}\varprojlim_{\mathcal{C}^{\mathrm{op}}} \mathcal{X}$$

as a groupoid. More precisely,

(1) The objects of it are of the form

$$L := \{(U, x_U); T_{f_{U,V}}\}_{U \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}, f_{U,V} \in \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(U,V)}$$

where $x_U \in \mathrm{Ob}\, F(U)$, $T_{f_{U,V}} \in \mathrm{Mor}_{\mathcal{X}^{\mathrm{op}}}((U, x_U), (V, x_V))$ such that $p^{\mathrm{op}}(T_{f_{U,V}}) = f_{U,V}$, $T_{f_{U_2,U_1}} \circ T_{f_{U_3,U_2}} = T_{f_{U_3,U_1}}$ for $f_{U_2,U_1} \circ f_{U_3,U_2} = f_{U_3,U_1}$ in $\mathcal{C}^{\mathrm{op}}$.

(2) The morphisms from L to $L' = \{(U, x'_U); T'_{f_{U,V}}\}$ are of the form $\{C_U\}_{U \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}}$, where $C_U \in \mathrm{Mor}_{\mathcal{X}^{\mathrm{op}}}((U, x_U), (U, x'_U))$ such that $T'_{f_{U,V}} \circ C_U = C_V \circ T_{f_{U,V}}$.

In Definition A.3, an object L determines and is determined by a functor $\mathscr{L} : \mathcal{C} \to \mathcal{X}$ sending U to (U, x_U) and $f^{\mathrm{op}}_{U,V}$ to $T^{\mathrm{op}}_{f_{U,V}}$ such that $p \circ \mathscr{L} = \mathrm{id}_{\mathcal{C}}$.

Let F_1, F_2, $(\mathcal{X}_1, p_1)$, $(\mathcal{X}_2, p_2)$, $\mathbf{F}$, and $\mathscr{F}$ be as in §A.1. Suppose that there is a 1-morphism $\pi : \mathcal{X}_2 \to \mathcal{X}_1$ (such that $p_1 \circ \pi = p_2$, and) such that $(\mathcal{X}_2, p)$ is isomorphic to the fibered category in groupoids over $\mathcal{X}_1$ associated with a functor $\pi^{presh} : \mathcal{X}_1^{\mathrm{op}} \to \mathsf{Groupoids}$. Denote by $\Phi : F_2 \to F_1$ the 1-morphism between presheaves corresponding to π according to Lemma A.2.

Lemma A.4. *In this setup, an object in $2\text{-}\varprojlim_{\mathcal{X}_1^{\mathrm{op}}} \mathcal{X}_2$ determines and is determined by a section $\mathscr{S} : \mathcal{X}_1 \to \mathcal{X}_2$ of $\pi : \mathcal{X}_2 \to \mathcal{X}_1$ in $1\text{-}\mathrm{Mor}_{\mathcal{C}}(\mathcal{X}_1, \mathcal{X}_2)$, which, from Lemma A.2, corresponds to a 1-morphism $\Psi : F_1 \to F_2$ such that $\Phi \circ \Psi = \mathrm{id}_{F_1}$. Moreover, the groupoid of such sections $\mathscr{S}$ (with morphisms among them defined by the natural transformations between functors from $\mathcal{X}_1$ to $\mathcal{X}_2$) is* isomorphic *to the groupoid $2\text{-}\varprojlim_{\mathcal{X}_1^{\mathrm{op}}} \mathcal{X}_2$.*

Proof. In the context of this lemma, we have $p_2 \circ \mathscr{L} = p_1 \circ \pi \circ \mathscr{L} = p_1$, which means that $\mathscr{L}$ is automatically a morphism in $1\text{-}\mathrm{Mor}_{\mathcal{C}}(\mathcal{X}_1, \mathcal{X}_2)$; note that every morphism is strongly Cartesian by [Sta25, Lem. 003V]. The first part of the lemma now follows from Lemma A.2. The isomorphism of groupoids follows from the construction in Definition A.3(2). □

Remark A.5. *By Lemma A.4, the three notions "an object in $2\text{-}\varprojlim_{\mathcal{X}_1^{\mathrm{op}}} \mathcal{X}_2$", "a 1-morphism section of $\pi : \mathcal{X}_2 \to \mathcal{X}_1$", and "a 1-morphism section of $\Phi : F_2 \to F_1$" can and should be used interchangeably in this paper.*

Assume further that $\mathcal{X}_1^{\lim} :=$

$$\underset{U \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}}{2\text{-}\lim} \varinjlim_{F_1(U)} F_1(U)$$

exists. If it exists, this is a groupoid since the inner colimits $\varinjlim_{F_1(U)} F_1(U)$ (that is, the colimit in the category $F_1(U)$) are groupoids, as they are unique up to unique isomorphisms by definition.

Denote by $\mathcal{X}_1'$ the full subcategory of $\mathcal{X}_1$ consisting of objects of the form $(U, \varinjlim_{F_1(U)} F_1(U))$. Note that there is also a projection $\ell : \mathcal{X}_1 \to \mathcal{X}_1'$ according to the definition of colimits, which makes natural inclusion $i : \mathcal{X}_1' \to \mathcal{X}_1$ a section of ℓ, i.e., $\ell \circ i = \mathrm{id}_{\mathcal{X}_1}$.

Lemma A.6. *With the assumptions above, there is a natural equivalence of groupoids*

$$2\text{-}\varprojlim_{\mathcal{X}_1^{\mathrm{op}}} \mathcal{X}_2 \cong 2\text{-}\varprojlim_{\mathcal{X}_1^{\prime,\mathrm{op}}} \mathcal{X}_2.$$

The second limit is formed for the projection $\ell \circ \pi : \mathcal{X}_2 \to \mathcal{X}_1'$.

Proof. The functor E_1 from the LHS to the RHS is the natural projection; the functor E_2 in the other direction is defined by pulling back objects of $\mathcal{X}_2$ over $\mathcal{X}_1' \subset \mathcal{X}_1$ via the morphisms defined by colimits (since $\pi : \mathcal{X}_2 \to \mathcal{X}_1$ is a fibered category by assumption). The fact that $E_2 \circ E_1 \simeq \mathrm{id}$(of LHS) is due to the universal property of strongly Cartesian morphisms and the assumption that $\mathcal{X}_2$ is fibered in groupoids: Write an object in LHS by $L = \{(x_1, x_2); T_{f_{x_1,x_1'}}\}$. Suppose that $x_1 \in \operatorname{Ob} F_1(U)$ maps to $x_1^* \in \operatorname{Ob} \varinjlim_{F_1(U)} F_1(U)$. Suppose that $p : x_2^* \to x_1^*$ is the object in the data of L. Then there is a (strongly Cartesian) lift $g : x_2' \to x_2^*$ of $f : x_1 \to x_1^*$. As $\mathcal{X}_2 \to \mathcal{X}_1$ is fibered in groupoids and there is a isomorphism $T_{f_{x_1,x_1^*}} : x_2 \to x_2^*$, there is a unique isomorphism $C_{x_1} : x_2 \to x_2'$.

The commutativity in Definition A.3(2) is obtained in a similar way. In fact, let y_1, y_1^*, y_2^*, y_2', and y_2 be the objects constructed in the same way for another $V \to U$ in $\mathcal{C}$. We have a diagram

(A.1)

$$\begin{tikzcd}
 & x_2 \arrow[dr] \arrow[drr] & & \\
y_2 \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[ddrr, bend left] & & x_2' \arrow[r] \arrow[d] & x_2^* \arrow[d] \\
 & y_2' \arrow[ur] \arrow[d] \arrow[dr] & x_1 \arrow[r] & x_1^* \\
 & y_1 \arrow[ur] \arrow[dr] & y_2^* \arrow[uur] \arrow[d] & \\
 & & y_1^*. \arrow[uur] &
\end{tikzcd}$$

To show the commutativity of the diagram formed by y_2, x_2, x_2' and y_2', it suffices to show the commutativity of y_2, x_2, x_2^* and y_2^*. This follows from the definition of 2-limits. □

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