EXAMPLES AND COUNTEREXAMPLES OF HODGE-TYPE LIFTINGS

SHENGKAI MAO AND PEIHANG WU

Abstract. This note serves as supplementary material to [MW26]. We compute the kernel of $\pi_1(G^{\mathrm{der}})_I \to \pi_1(G)_I$ for the Hodge-type liftings of [Del79] and [KPZ24]. We provide some examples and counterexamples to justify the necessity of finding accessible Hodge-type liftings.

1. Introduction

This note is not intended for publication. We work out several examples of Shimura data of type $D^{\mathbb{H}}$ and present some counterexamples to the injectivity of the map $\pi_1(G^{\mathrm{der}})_I \to \pi_1(G)_I$.

The main subtlety is that a naive application of the construction in [Del79] and [KPZ24] might not be good enough to obtain injectivity. Some counterexamples will be given in §2.2, the main computation will be given in Proposition 3.7, and the modified construction will be given in §4. The readers are suggested to jump directly to the corresponding places. Some interesting lemmas are recorded along the way.

The notation system is the same as [MW26].

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2. Preliminaries

2.1. **Deligne's construction and refinements.** Let (G, X) be a Shimura datum. By the axioms of (G, X) being a Shimura datum, G^{ad} can be written as $G^{\mathrm{ad}} = \prod_i R_{F_i/\mathbb{Q}} H_i^{\mathrm{ad}}$ for totally real fields F_i over $\mathbb{Q}$; the groups H_i^{ad} are absolutely simple adjoint reductive groups over F_i. See [Del79, 2.3.4] and the proof of [Mil11, Thm. 3.13].

Proposition 2.1 (See [Del79] and [KPZ24]). *Given an abelian-type Shimura datum (G_2, X_2), by [Del79, Prop. 2.3.10] (see also [KPZ24, 7.2]), one can always associate with it a Hodge-type Shimura datum (G, X) that satisfies the following properties:*

- *$G^{\mathrm{der}} = \prod_{i\in J} R_{F_i/\mathbb{Q}} H_i$, for F_i a totally real field over $\mathbb{Q}$, H_i an absolutely almost simple semisimple group over F_i, and $H_i = H_i^{\mathrm{sc}}$ if H_i is of type $A, B, C, D^{\mathbb{R}}$, $H_i = H_i^{\mathrm{sc}}/\mu_2$ if H_i is of type $D^{\mathbb{H}}$. In the last case, for any embedding $F_i \hookrightarrow \mathbb{R}$, $H_{i,\mathbb{R}}$ is isomorphic to SO^*_{2n+2} or is compact. The index J is a finite set with a partition $J = J_c \coprod J_{nc}$, where J_c (resp. J_{nc}) is defined such that, for each $i \in J_c$ (resp. $i \in J_{nc}$), $R_{F_i/\mathbb{Q}} H_i^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic (resp. $\mathbb{Q}$-isotropic).*
- *$G = G^{\mathrm{der}} \cdot Z^c \cdot \mathbb{G}_m$. One can choose Z^c such that Z^c is $\mathbb{Q}$-anisotropic, and $Z^c = \prod_{i\in J_c} R_{F_i/\mathbb{Q}} R^{(1)}_{K_i/F_i} \mathbb{G}_{\mathrm{m}}$, where K_i are CM extensions of F_i. Note that our choice of Z^c does not have factors when $i \in J_{nc}$.*
- *The intersection of $\mathbb{G}_m$ and $G^{\mathrm{der}} \cdot Z^c$ is always μ_2. In the type $D^{\mathbb{H}}$ case, the intersection of G^{der} and Z^c is $\prod_{i\in J_c} R_{F_i/\mathbb{Q}} \mu_2$.*
- *The reflex field is contained in $\mathbb{E}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cdot \prod_{i\in J_c} K_i$. The field extensions K_i can be taken such that the field extension $\mathbb{E}(G_2, X_2) \cdot K_i/\mathbb{E}(G_2, X_2)$ splits completely at every place over p. In fact, $K_i = F_i \cdot K_i'$, where K_i' is a totally imaginary quadratic extension of $\mathbb{Q}$.*

Remark 2.2. *In [KPZ24], one chooses $Z^c = \prod_{i\in J} R_{F_i/\mathbb{Q}} R^{(1)}_{K_i/F_i}\mathbb{G}_{\mathrm{m}}$ to make sure that $Z(G)$ is connected. With this change, the kernel in Proposition 3.7 is larger. Hence, we still get some counterexamples to the injectivity. So we can choose a smaller group here.*

2.2. **Forms of** SO **and** GSO.

Example 2.3. *We recall the coordinate system in [YZZ26, §2.1] (cf. [PRS10, §2.7]). Let K be a p-adic local field and $t \in \mathcal{O}_K$ be a uniformizer (with $\mathrm{val}_K(t) = 1$). Let $L = K(u)$ be a quadratic extension with $u^2 = -t$ and $\mathrm{val}_L(u) = 1$. Let V be a $2(n+1)$-dimensional K-vector space with an ordered basis $e_1, \dots, e_{2n}, f_1, f_2$. Let ϕ be a symmetric K-bilinear form on V, whose matrix with respect to this basis is*

$$\phi = \begin{pmatrix} H_{2n} & \\ & \begin{pmatrix} t & \\ & 1 \end{pmatrix} \end{pmatrix}, \tag{2.1}$$

where H_{2n} is the anti-diagonal unit matrix of size $2n$. $\tilde{G} := \mathrm{GO}(V,\phi)$ is quasi-split over K. Note that $\tilde{G}$ is non-connected, $\tilde{G} = \tilde{G}^\circ \sqcup \tau\tilde{G}^\circ$. Let $\eta : \tilde{G} \to \mathbb{G}_{\mathrm{m}}$ be the similitude character, then $\tilde{G}^\circ = \{g \in \tilde{G} |\ \eta(g)^{n+1} = \det(g)\}$, $\tau = \mathrm{diag}(\mathrm{id}_{2n}, -1, 1)$. We denote $G := \mathrm{GSO}(V,\phi) := \tilde{G}^\circ$, and $\eta : G \to \mathbb{G}_{\mathrm{m}}$ the restriction of the similitude character.

- *Fix the standard maximal K-torus T of G whose set of R-points is*
$$\left\{ \mathrm{diag}\left(x_1, \dots, x_{2n}, \begin{pmatrix} y_1 & y_2 \\ -ty_2 & y_1 \end{pmatrix}\right) \in \mathrm{GL}_{2n+2}(R) \;\middle|\; \begin{array}{l} x_1x_{2n} = \cdots = x_nx_{n+1} = y_1^2 + ty_2^2, \\ x_1, \dots, x_{2n}, y_1, y_2 \in R \end{array} \right\}.$$
- *Choose*
$$\varphi : (\mathbb{G}_{\mathrm{m}})_L \times (\mathbb{G}_{\mathrm{m}})_L \xrightarrow{\sim} \left\{ A \in \mathrm{GL}_{2,L} \;\middle|\; A^t \begin{pmatrix} t & \\ & 1 \end{pmatrix} A = \det(A) \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right\},$$
$$T(L) = \left\{ \mathrm{diag}(x_1, \dots, x_{2n}, \varphi(z_1, z_2)) \in \mathrm{GL}_{2n+2}(L) \;\middle|\; \begin{array}{l} x_1x_{2n} = \cdots = x_nx_{n+1} = z_1z_2, \\ x_1, \dots, x_{2n}, z_1, z_2 \in L \end{array} \right\}.$$
- *We have the isomorphism $T(L)/T(\mathcal{O}_L) \xrightarrow{\sim} X_*(T_L) \subset \mathbb{Z}^{2n+2}$:*
$$\mathrm{diag}(x_1, \dots, x_{2n}, \varphi(z_1, z_2)) \mapsto (\mathrm{val}_L(x_1), \dots, \mathrm{val}_L(x_{2n}), \mathrm{val}_L(z_1), \mathrm{val}_L(z_2)).$$
- *Under the condition*
$$\mathrm{val}_L(x_1) + \mathrm{val}_L(x_{2n}) = \cdots = \mathrm{val}_L(x_n) + \mathrm{val}_L(x_{n+1}) = \mathrm{val}_L(z_1) + \mathrm{val}_L(z_2)$$
in $X_*(T_L)$, we have an isomorphism $X_*(T) \xrightarrow{\sim} \mathbb{Z}^{n+2}$
$$(a_1, \dots, a_n, a_{n+1}, \dots a_{2n}, b_1, b_2) \mapsto (a_1, \dots, a_n, b_1, b_2).$$
- *$\delta \in \Delta = \mathrm{Gal}(L/K)$ acts on $X_*(T) \cong \mathbb{Z}^{n+2}$ as*
$$(a_1, \dots, a_n, b_1, b_2) \mapsto (a_1, \dots, a_n, b_2, b_1).$$
In particular, $X_*(T)_\Delta = \mathbb{Z}^n \oplus \mathbb{Z}^2/\mathbb{Z}(1,-1) \cong \mathbb{Z}^{n+1}$; here $\mathbb{Z}^2/\mathbb{Z}(1,-1) \xrightarrow{\sim} \mathbb{Z} : (b_1, b_2) \mapsto b_1 + b_2$.
- *The coroot lattice $Q^\vee \subset X_*(T)$ is*
$$\left\{ (a_1, \dots, a_n, b_1, b_2) \in \mathbb{Z}^{n+2} \;\middle|\; a_1 + \cdots + a_n + b_1 \text{ is even, and } b_1 + b_2 = 0 \right\}.$$
In particular, the image of $Q^\vee_\Delta$ in $X_*(T)_\Delta \cong \mathbb{Z}^{n+1}$ can be identified as $\{(a_1, \dots, a_n, 0)\} \cong \mathbb{Z}^n$, and $\pi_1(G)_\Delta = X_*(T)_\Delta/Q^\vee_\Delta \cong \mathbb{Z}$.

Next, we consider the derived group $G^{\mathrm{der}} = \mathrm{SO}(V,\phi) = \ker\eta$.

- *Let $T^{\mathrm{der}} \subset T$ be the maximal torus of G^{der} with*
$$T^{\mathrm{der}}(L) = \{\mathrm{diag}(x_1, \dots, x_{2n}, \varphi(z_1, z_2)) \in T(L) \mid z_1 z_2 = 1\},$$
$$X_*(T^{\mathrm{der}}) = \{(a_1, \dots, a_n, b_1, b_2) \in X_*(T) \mid b_1 + b_2 = 0\}.$$
- *Inside $X_*(T) \cong \mathbb{Z}^{n+2}$, we view*
$$X_*(T^{\mathrm{der}}) \cong \mathbb{Z}^n \oplus \mathbb{Z}(1,-1), \quad X_*(T^{\mathrm{der}})_\Delta \cong \mathbb{Z}^n \oplus \mathbb{Z}(1,-1)/\mathbb{Z}(2,-2) \cong \mathbb{Z}^n \oplus \mathbb{Z}/2\mathbb{Z}.$$
Here we take $\mathbb{Z}(1,-1)/\mathbb{Z}(2,-2) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$, $(b,-b) \to \bar{b} \mod 2$.
- *Inside $X_*(T^{\mathrm{der}})_\Delta$, the image of $Q^\vee_\Delta$ is*
$$\left\{(a_1, \dots, a_n, \overline{\Sigma_i a_i})\right\} \cong \mathbb{Z}^n.$$
In particular, $\pi_1(G^{\mathrm{der}})_\Delta = X_*(T^{\mathrm{der}})_\Delta / Q^\vee_\Delta \cong \mathbb{Z}/2\mathbb{Z}$. The unique nontrivial element is the image of $(0, \dots, 0, 1, -1) \in X_*(T^{\mathrm{der}})$.

We see that $\pi_1(G^{\mathrm{der}})_\Delta \to \pi_1(G)_\Delta$ is $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, it is not injective, even if $p > 2$.

Finally, let us compute $H_1(\Delta, \pi_1(G))$. Consider the surjection
$$\pi : X_*(T) \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad (a_1, \dots, a_n, b_1, b_2) \mapsto (b_1 + b_2, \overline{a_1 + \cdots + a_n + b_1}).$$
We have $\ker \pi = Q^\vee$; then $\pi_1(G) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, and $\delta \in \Delta$ acts on $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ as $(u, \epsilon) \mapsto (u, \epsilon + \bar{u})$. Recall that $H_1(\Delta, \pi_1(G)) = \ker(\delta - 1)/\operatorname{Im}(1 + \delta)$. Given $(u, \epsilon) \in \pi_1(G)$, $(\delta - 1)(u, \epsilon) = (0, \bar{u})$, and $(1 + \delta)(u, \epsilon) = (2u, \bar{u})$. In particular, $H_1(\Delta, \pi_1(G)) = \mathbb{Z}/2\mathbb{Z}$.

Remark 2.4. *Any reductive group over $\breve{\mathbb{Q}}_p$ is quasi-split and moreover residually split. When $p > 2$, any quasi-split non-split form ϕ over $\breve{\mathbb{Q}}_p$ has standard basis expressed in Example 2.3 (see* [PRS10, §2.7]*). When $p = 2$, ϕ might not have expression (2.1) due to the complexity of $\breve{\mathbb{Q}}_2^\times / (\breve{\mathbb{Q}}_2^\times)^2$. Nevertheless, in Tit's classification* [Tit79, §4.2, 4.4]*, any residually split absolutely almost simple group $\mathrm{SO}(V, q)$ in $\dim V = r$ variables with Witt index r' satisfies $r = 2r'$ or $r = 2r' + 2$. When $r = 2r'$, the group is split. When $r = 2r' + 2$, V has an orthogonal decomposition by a $2r'$-dimensional hyperbolic subspace and a 2-dimensional anisotropic subspace, where the quadratic form q on the latter space can be expressed as the Norm map (see* [Tit79, §1.16]*). Under this decomposition, the standard maximal torus can still be written as $T \cong \mathbb{G}_{\mathrm{m}}^n \times R_{L/K}\mathbb{G}_{\mathrm{m}}$, under which $T^{\mathrm{der}} \cong \mathbb{G}_{\mathrm{m}}^n \times R^{(1)}_{L/K}\mathbb{G}_{\mathrm{m}}$. Using the standard coordinates of $\mathbb{G}_{\mathrm{m}}^n \times R_{L/K}\mathbb{G}_{\mathrm{m}}$, the coroots $Q^\vee$ has the same form as in Example 2.3, and the Galois action of Δ switches the last two coordinates in $X_*(T)$; the calculation remains the same.*

Remark 2.5. *Let $K = \mathbb{Q}_p$. We start with a group as in Example 2.3 or* [KPZ24, 6.2.2(b)]*; we can choose a quadratic form ϕ there such that it can be globalized to a form $\boldsymbol{\phi}$ over $\mathbb{Q}$ and the group $\mathbf{G} = \mathrm{GSO}(V, \boldsymbol{\phi})$ splits over a totally imaginary quadratic extension. Then $\mathbf{G}$ can be upgraded to a Hodge-type Shimura datum by* [Del79, 1.3.9 and 2.3.13]*. Then the computation above implies that $\pi_1(\mathbf{G}^{\mathrm{der}}_{\mathbb{Q}_p})_I \to \pi_1(\mathbf{G}_{\mathbb{Q}_p})_I$ is not injective, even if $p > 2$! This demonstrates how the type $D^{\mathbb{H}}$ complicates the situation, and also justifies the necessity of having* [MW26, Def. 6.16] *and the computations below.*

Example 2.6. *Keeping the notation from Example 2.3, we consider the split form of GO. To be compatible with the coordinate system of the quasi-split form, let $\tilde{G} = \mathrm{GO}(V, \phi)$, and $G := \mathrm{GSO}(V, \phi) = \tilde{G}^\circ$, where V has ordered basis $e_1, \dots, e_{2n}, f_1, f_2$, and ϕ has the matrix with respect to this basis:*
$$\phi = \begin{pmatrix} H_{2n} & \\ & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{pmatrix}.$$

- *Fix the standard maximal K-split torus T of G whose set of R-points is*

$$\left\{ \operatorname{diag}(x_1,\dots,x_{2n},z_1,z_2)\in \mathrm{GL}_{2n+2}(R) \;\middle|\; \begin{array}{l} x_1x_{2n}=\cdots=x_nx_{n+1}=z_1z_2,\\ x_1,\dots,x_{2n},z_1,z_2\in R \end{array}\right\}.$$

- *Descriptions of $X_*(T)$, $Q^\vee$, T^{der}, $X_*(T^{\mathrm{der}})$ are the same as those in Example 2.3.*
- *Δ acts trivially on everything.*
- *Under $\pi : X_*(T)\to \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$, the sub-lattice $X_*(T^{\mathrm{der}})$ maps to $\mathbb{Z}/2\mathbb{Z}\oplus 0$. In particular $\pi_1(G^{\mathrm{der}})=\mathbb{Z}/2\mathbb{Z}$.*
- *Under the above identification, $\pi_1(G^{\mathrm{der}})\to\pi_1(G)$ is $\mathbb{Z}/2\mathbb{Z}\xrightarrow{(\mathrm{id},0)}\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$.*

3. Calculation of Hodge-lifting

Convention 3.1. *Let H and G be two connected reductive groups over $\mathbb{Q}$ or $\breve{\mathbb{Q}}_p$ with a map $f : H\to G$. Denote* $\ker(f:\pi_1(H_{\breve{\mathbb{Q}}_p})_I\to\pi_1(G_{\breve{\mathbb{Q}}_p})_I)$ *by $\pi(H,G)$.*

Given $f : H\to G$ and $g : G\to G'$, we have, by definition, an inclusion $\pi(H,G)\subset\pi(H,G')$.

3.1. Reduction to type $D^{\mathbb{H}}$.

We factor G^{der} into $G_1^{\mathrm{der}}\times G_2^{\mathrm{der}}$, where G_1^{der} is a group with types A-, B-, C- and $D^{\mathbb{R}}$-factors, and where G_2^{der} is a group with only type $D^{\mathbb{H}}$-factors. Let $G_1^{\mathrm{der}}\to G_1$ and $G_2^{\mathrm{der}}\to G_2$ be the objects obtained from the Deligne's construction. Let $G := G_1\times_{\eta_1,\mathbb{G}_{\mathrm{m}},\eta_2} G_2\subset G_1\times G_2$, where $\eta_i : G_i\to\mathbb{G}_{\mathrm{m}}$ are the similitude characters for $i=1,2$. Let $\eta : G\to\mathbb{G}_{\mathrm{m}}$ be the similitude character. By construction, G_1^{der} is simply connected, i.e., $\pi_1(G_1^{\mathrm{der}})=0$; thus, $\pi_1(G^{\mathrm{der}})=0\oplus\pi_1(G_2^{\mathrm{der}})$. We have a natural projection $G\to G_2$, but we do not have a section $G_2\to G$ in general.

Lemma 3.2. *With the conventions above, $\pi(G^{\mathrm{der}},G)=\pi(G_2^{\mathrm{der}},G_2)$.*

Proof. Consider the exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r, "\Delta"] \arrow[d, "\eta"'] & G_1\times G_2 \arrow[r, "\eta_1\eta_2^{-1}"] \arrow[d, "(\eta_1{,}\eta_2)"'] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r, "\Delta"] & \mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}} \arrow[r, "\mathrm{pr}_1\,\mathrm{pr}_2^{-1}"] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1.
\end{tikzcd}$$

Note that the lower line has a section $s:\mathbb{G}_{\mathrm{m}}\to\mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}}$, $x\mapsto(x,1)$ to the projection $\mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}}\to\mathbb{G}_{\mathrm{m}}$, $(x,y)\mapsto xy^{-1}$. Taking the long exact sequences, we have:

$$\begin{tikzcd}
 & \pi_1(G^{\mathrm{der}})_I \arrow[r, equal, "="] \arrow[d, "i"'] & 0\oplus\pi_1(G_2^{\mathrm{der}})_I \arrow[d, "0\times i_2"'] & \\
H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r, "\iota"] \arrow[d, equal, "="'] & \pi_1(G)_I \arrow[r, "j"] \arrow[d] & \pi_1(G_1)_I\oplus\pi_1(G_2)_I \arrow[r] \arrow[d] & \pi_1(\mathbb{G}_{\mathrm{m}})_I \arrow[d, equal, "="] \\
H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r, "\bar\iota=0"] & \pi_1(\mathbb{G}_{\mathrm{m}})_I \arrow[r] & \pi_1(\mathbb{G}_{\mathrm{m}})_I\oplus\pi_1(\mathbb{G}_{\mathrm{m}})_I \arrow[r] & \pi_1(\mathbb{G}_{\mathrm{m}})_I.
\end{tikzcd}$$

The long exact sequence are the rows.

It suffices to show j is an injection. This is equivalent to showing that $\iota=0$. Consider more terms in the long exact sequences:

$$\begin{tikzcd}
H_1(I,\pi_1(G_1))\oplus H_1(I,\pi_1(G_2)) \arrow[r, "(\pi_1{,}\pi_2)"] \arrow[d, "(p_1{,}p_2)"'] & H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r, "\iota"] \arrow[d, equal, "="'] & \pi_1(G)_I \arrow[d] \\
H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}}))\oplus H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r, "(\bar\pi_1{,}\bar\pi_2)"] & H_1(I,\pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r, "\bar\iota=0"] & \pi_1(\mathbb{G}_{\mathrm{m}})_I.
\end{tikzcd}$$

Note that $\iota = 0$ is equivalent to (π_1, π_2) being surjective. We claim that π_1 is already surjective.

On one hand, we show that p_1 is surjective, which is equivalent to $\pi_1(G_1^{\mathrm{der}} \cdot Z_1^c)_I \to \pi_1(G_1)_I$ being injective. Here $\ker \eta_1 = G_1^{\mathrm{der}} \cdot Z_1^c$ by construction. Consider $G_1^{\mathrm{der}} \cdot Z_1^c \to G_1^{ab} = G_1/G_1^{\mathrm{der}}$. Since $\pi_1(G_1^{\mathrm{der}}) = 0$, and $\mu_2 = \mathbb{G}_{\mathrm{m}} \cap (G_1^{\mathrm{der}} \cdot Z_1^c)$ is contained in $G_1^{\mathrm{der}} \cap Z_1^c$, there is a section $\mathbb{G}_{\mathrm{m}}/\mu_2 \to G_1^{ab}$ in the exact sequence $1 \to (G_1^{\mathrm{der}} \cdot Z_1^c)^{ab} \to G_1^{ab} \to \mathbb{G}_{\mathrm{m}}/\mu_2 \to 1$; thus $\pi_1((G_1^{\mathrm{der}} \cdot Z_1^c)^{ab})_I \to \pi_1(G_1^{ab})_I$ is injective, we are done.

On the other hand, the section s induces a section of $\bar{\pi}_1$; in particular, $\bar{\pi}_1$ is an isomorphism. Since $\pi_1 = \bar{\pi}_1 \circ p_1$, π_1 is surjective. The inclusion in the lemma follows from the paragraph under Convention 3.1. □

3.2. **Further reductions.** We introduce some building blocks of the main calculation.

Lemma 3.3. *Let G_1 and G_2 be reductive groups over $\breve{\mathbb{Q}}_p$ whose centers contain μ_2. Set*

$$H_1 := (\mathbb{G}_{\mathrm{m}} \times G_1)/\mu_2, \quad H_2 := (\mathbb{G}_{\mathrm{m}} \times G_2)/\mu_2, \quad H := (\mathbb{G}_{\mathrm{m}} \times G_1 \times G_2)/\mu_2.$$

(1) Assume that $\pi_1(G_2)_I \to \pi_1(H_2)_I$ is injective. Then

$$\pi(G_1 \times G_2, H) = \pi(G_1, H_1) \times 0.$$

(2) Assume that $\pi_1(G_1)_I \to \pi_1(H_1)_I$ and $\pi_1(G_2)_I \to \pi_1(H_2)_I$ are both injective. Then $\pi_1(G_1 \times G_2)_I \to \pi_1(H)_I$ is injective.

Proof. (1) This can be read from the long exact sequences associated with the commutative diagram with exact vertical rows:

$$\begin{tikzcd}
G_1 \arrow[r, dash, "="] \arrow[d] & G_1 \arrow[r] \arrow[d] & H_1 \arrow[d] \\
G_1 \times G_2 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H_1 \times H_2 \arrow[d] \\
G_2 \arrow[r] & H_2 \arrow[r, dash, "="] & H_2.
\end{tikzcd}$$

The $(\subset)$ direction is easy. Now, let $[a] \in \ker(\pi_1(G_1)_I \to \pi_1(H_1)_I)$ and let $[\tilde{a}] \in \pi_1(H)_I$ be the image of $[a]$. We can lift $[\tilde{a}]$ to $[b] \in \pi_1(G_2)_I$ using the exact sequence $\pi_1(G_2)_I \to \pi_1(H)_I \to \pi_1(H_1)_I \to 1$. The image of $[b]$ in $\pi_1(H_1 \times H_2)_I = \pi_1(H_1)_I \times \pi_1(H_2)_I$ is the image of $[\tilde{a}]$; thus, it is the image of $[a]$, which is trivial. Since $\pi_1(G_2)_I \hookrightarrow \pi_1(H_2)_I$, $[b]$ itself is trivial; we get the $(\supset)$ direction.

(2) This is because $G_1 \times G_2 \to H_1 \times H_2$ factors through H.

□

Lemma 3.4. *Keep the notation from Lemma 3.3. For $i = 1, 2$, let $\widetilde{G}_i$ be a reductive group over $\breve{\mathbb{Q}}_p$, with an embedding $G_i \to \widetilde{G}_i$ such that $\pi_1(G_i)_I \to \pi_1(\widetilde{G}_i)_I$ is an isomorphism. Let $\widetilde{H} = (\mathbb{G}_{\mathrm{m}} \times \widetilde{G}_1 \times \widetilde{G}_2)/\mu_2$. Then $\pi(G_1 \times G_2, H) = \pi(\widetilde{G}_1 \times \widetilde{G}_2, \widetilde{H})$.*

Proof. We have exact sequences

$$1 \to G_1 \times G_2 \to H \to \mathbb{G}_{\mathrm{m}} \to 1, \quad 1 \to \widetilde{G}_1 \times \widetilde{G}_2 \to \widetilde{H} \to \mathbb{G}_{\mathrm{m}} \to 1,$$

where the surjections are given by $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, $x \mapsto x^2$. Consider the long exact sequences:

$$\begin{tikzcd}
H_1(I, \pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r] \arrow[d, dash, "="] & \pi_1(G_1)_I \oplus \pi_1(G_2)_I \arrow[r] \arrow[d, "\cong"] & \pi_1(H)_I \arrow[r] \arrow[d] & \pi_1(\mathbb{G}_{\mathrm{m}})_I \arrow[d, "="] \\
H_1(I, \pi_1(\mathbb{G}_{\mathrm{m}})) \arrow[r] & \pi_1(\widetilde{G}_1)_I \oplus \pi_1(\widetilde{G}_2)_I \arrow[r] & \pi_1(\widetilde{H})_I \arrow[r] & \pi_1(\mathbb{G}_{\mathrm{m}})_I,
\end{tikzcd}$$

By Five Lemma, $\pi_1(H)_I \to \pi_1(\widetilde{H})_I$ is an isomorphism. □

Lemma 3.5. *Let $G := \mathrm{GSO}_{2n+2}$ be either split or quasi-split over $\breve{\mathbb{Q}}_p$. Set $H = (\mathbb{G}_{\mathrm{m}} \times \mathrm{GSO}_{2n+2})/\mu_2$. Then $\pi_1(G)_I \to \pi_1(H)_I$ is injective.*

Proof. Consider the exact sequence $1 \to G \to H \to \mathbb{G}_{\mathrm{m}} \to 1$ and the induced long exact sequence

$$H_1(I, \pi_1(\mathbb{G}_{\mathrm{m}})) \to \pi_1(G)_I \to \pi_1(H)_I \to \pi_1(\mathbb{G}_{\mathrm{m}})_I \to 1.$$

When G is split, I acts trivially on everything, $(\pi_1(G)_I \to \pi_1(H)_I) = (\pi_1(G) \to \pi_1(H))$ is injective. When G is quasi-split, in the calculation 2.3, the I-action on the modules factors through $\Delta = \langle \delta \rangle$, $\delta^2 = 1$. Since δ acts on $\mathbb{G}_{\mathrm{m}}$ trivially, $H_1(\Delta, \pi_1(\mathbb{G}_{\mathrm{m}})) = \ker(\delta - 1)/\mathrm{Im}(\delta + 1) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(G)_\Delta = \mathbb{Z}$. Thus, $H_1(\Delta, \pi_1(\mathbb{G}_{\mathrm{m}})) \to \pi_1(G)_\Delta$ is a trivial map, $(\pi_1(G)_I \to \pi_1(H)_I) = (\pi_1(G)_\Delta \to \pi_1(H)_\Delta)$ is injective. ☐

Lemma 3.6. *Let $i \in J = \{1, 2, \dots, n\}$, and $G_i := \mathrm{SO}_{2n_i+2}$ be a quasi-split non-split form over $\breve{\mathbb{Q}}_p$. By Example 2.3 and Remark 2.4, $\pi_1(G_i)_I = \mathbb{Z}/2\mathbb{Z}$. Let $G = ((\prod_{i\in J} G_i) \times \mathbb{G}_{\mathrm{m}})/\mu_2$, where μ_2 embeds diagonally on every factor. Then the embedding $\prod_{i\in J} G_i \to G$ induces*

$$\ker\left(\prod_{i\in J} \pi_1(G_i)_I \to \pi_1(G)_I\right) = \mathbb{Z}/2\mathbb{Z} \overset{\Delta}{\hookrightarrow} \prod_{i\in J} \mathbb{Z}/2\mathbb{Z} = \prod_{i\in J} \pi_1(G_i)_I.$$

Proof. Write $G_i \hookrightarrow H_i = \mathrm{GSO}_{2n_i+2}$, and let $\eta_i : H_i \to \mathbb{G}_{\mathrm{m}}$ be the similitude character. Then $G = H_1 \times_{\eta_1, \mathbb{G}_{\mathrm{m}}, \eta_2} \cdots \times_{\eta_{n-1}, \mathbb{G}_{\mathrm{m}}, \eta_n} H_n \subset H := H_1 \times \cdots \times H_n$, $G^{\mathrm{der}} = \prod_{i\in J} G_i$. We use the coordinates in Example 2.3. Let $T_i \subset H_i$ be the standard maximal torus, and $T_i^{\mathrm{der}} := T_i \cap G_i \subset G_i$. Then $T = ((\prod_{i\in J} T_i) \times \mathbb{G}_{\mathrm{m}})/\mu_2$ (resp. $T^{\mathrm{der}} = \prod_{i\in J} T_i^{\mathrm{der}}$, $T_H = \prod_{i\in J} T_i$) is the standard maximal torus of G (resp. G^{der}, H). The coroot $Q^\vee$ of G is $\prod_i Q_i^\vee$. Fix the coordinates:

$$X_*(T_H) = \prod_{i\in J} \left\{(\underline{a}_i, b_{i1}, b_{i2}) \in \mathbb{Z}^{n_i+2}\right\} \cong \prod_{i\in J} \mathbb{Z}^{n_i+2},$$

where $\underline{a}_i = (a_{i1}, \dots, a_{in_i}) \in \mathbb{Z}^{n_i}$. The I-action factors through $\Delta = \langle \delta \rangle$, $\delta^2 = 1$, and δ acts on $X_*(T_H)$ as:

$$\prod_{i\in J} (\underline{a}_i, b_{i1}, b_{i2}) \mapsto \prod_{i\in J} (\underline{a}_i, b_{i2}, b_{i1}).$$

Under the coordinates above, we have

$$X_*(T) = \left\{ \prod_{i\in J} (\underline{a}_i, b_{i1}, b_{i2}) \in X_*(T_H) \,\middle|\, b_{i1} + b_{i2} = b_{j1} + b_{j2} \textit{ for all } i, j \in J \right\},$$

$$X_*(T^{\mathrm{der}}) = \left\{ \prod_{i\in J} (\underline{a}_i, b_{i1}, b_{i2}) \in X_*(T_H) \,\middle|\, b_{i1} + b_{i2} = 0, \forall i \in J \right\},$$

$$Q^\vee = \left\{ \prod_{i\in J} (\underline{a}_i, b_{i1}, b_{i2}) \in X_*(T_H) \,\middle|\, \underline{a}_i + b_{i1} \textit{ are even }, b_{i1} + b_{i2} = 0, \forall i \in J \right\}.$$

Fixing a realization of I-modules $X_*(T) \cong \prod_{i\in J} \mathbb{Z}^{n_i+1} \times \mathbb{Z}$ with assignments

$$\prod_{i\in J} (\underline{a}_i, b_{i1}, b_{i2}) \mapsto \left(\left(\prod_{i\in J} (\underline{a}_i, b_{i1}) \right), b_{11} + b_{12} \right),$$

the action of δ is

$$\delta : \left(\left(\prod_{i\in J} (\underline{a}_i, b_i) \right), c \right) \mapsto \left(\left(\prod_{i\in J} (\underline{a}_i, c - b_i) \right), c \right).$$

Let $e_{ij} = (0, \dots, 1, \dots 0) \in \mathbb{Z}^{n_i+1}$ be the j-th factor ($j \in J_i := \{1, \dots, n_i\}$), $f_i = (0, \dots, 0, 1) \in \mathbb{Z}^{n_i+1}$, and $k = (\prod_{i\in J} \underline{0}, 1) \in \prod_{i\in J} \mathbb{Z}^{n_i+1} \times \mathbb{Z}$ be the last coordinate. We have $\delta(e_{ij}) = e_{ij}$,

$\delta(f_i) = -f_i$, $\delta(k) = (\sum_{i\in J} f_i) + k$. Then $(\delta - 1)X_*(T)$ is generated by $\{2f_i\}_{i\in J}$ and $\Sigma_{i\in J} f_i$. We can identify

$$X_*(T)_I = \prod_{i\in J} \mathbb{Z}^{n_i} \times \prod_{i\in J}{}^{'} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}, \qquad \left(\left(\prod_{i\in J}(\underline{a}_i, b_i)\right), c\right) \mapsto \left(\prod_{i\in J} \underline{a}_i, \prod_{i\in J} \bar{b}_i, c\right),$$

where $\bar{*}$ means $*$ modulo 2, and $\prod'_{i\in J} \mathbb{Z}/2\mathbb{Z}$ is the quotient of $\prod_{i\in J} \mathbb{Z}/2\mathbb{Z}$ modulo the diagonal $\mathbb{Z}/2\mathbb{Z}$. Inside $X_*(T)_I$, the image of $Q^\vee$ is generated by $\langle e_{ij} + f_i\rangle_{i\in J, j\in J_i}$, and is free of rank $\sum_i n_i$;

$$\pi_1(G)_I = X_*(T)_I/Q_I^\vee \cong \prod_{i\in J}{}^{'} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}, \qquad \left(\prod_{i\in J} \underline{a}_i, \prod_{i\in J} \bar{b}_i, c\right) \mapsto \left(\prod_{i\in J} \bar{b}_i, c\right).$$

Since the unique nontrivial element $\pi_1(G_i)_I = X_*(T_i^{\mathrm{der}})_I/(Q_i^\vee)_I \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the image of $(\underline{0}, b_{i1}, -b_{i1}) \in X_*(T_i^{\mathrm{der}})$ with odd b_{i1}, we are done. □

3.3. **Main calculation.** Let (G_2, X_2) be an abelian-type Shimura datum and (G, X) be the associated Hodge-type Shimura datum defined using Deligne's construction as in Proposition 2.1.

Write $G^{\mathrm{der}} = \prod_{i\in J} R_{F_i/\mathbb{Q}} H_i$, and let J_c, J_{nc} be the sub-index sets defined as before. Let $J^1 \subset J$ (resp. $J^2 \subset J$) be the sub-index where H_i is of type $D^{\mathbb{H}}$ (resp. A, B, C, or $D^{\mathbb{R}}$). Let $J_c^t = J_c \cap J^t$, $J_{nc}^t = J_{nc} \cap J^t$, for $t = 1, 2$. After the base-change to $\breve{\mathbb{Q}}_p$, we have

$$G^{\mathrm{der}}_{\breve{\mathbb{Q}}_p} = \Big(\prod_{i\in J_c^1, j\in J_i^{ss}} \times \prod_{i\in J_c^1, j\in J_i^{qs}} \times \prod_{i\in J_{nc}^1, j\in J_i^{ss}} \times \prod_{i\in J_{nc}^1, j\in J_i^{qs}} \times \prod_{i\in J^2, j\in J_i}\Big) R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij}.$$

Here J_i is the index set of $(F_i \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p = \prod_{j\in J_i} F_{ij}$, F_{ij} are field extensions of $\breve{\mathbb{Q}}_p$, J_i^{ss} (resp. J_i^{qs}) are the sub-index where the F_{ij}-group H_{ij} is split (resp. quasi-split non-split). Note that every reductive over $\breve{\mathbb{Q}}_p$ is always quasi-split, $J_i = J_i^{ss} \sqcup J_i^{qs}$.

When $i \in J^2$, H_i are simply-connected over F_i, thus $\pi_1(H_{ij}) = 0$ for all $j \in J_i$. When $i \in J^1$, H_i is of type $D^{\mathbb{H}}$, any F_{ij}-group H_{ij} is SO_{2n_i+2} for some positive n_i, either split or quasi-split non-split. Let $I_{ij} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{ij})$. By Shapiro's lemma, $\pi_1(R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij})_I = \pi_1(H_{ij})_{I_{ij}} = \mathbb{Z}/2\mathbb{Z}$ for any $j \in J_i$. See Examples 2.3 and 2.6, and Remark 2.4.

Proposition 3.7. *In this case, $\pi(G^{\mathrm{der}}, G)$ is*

$$\langle 0\rangle \times \prod_{i\in J_c^1, j\in J_i^{qs}} \mathbb{Z}/2\mathbb{Z} \times \langle 0\rangle \times \mathbb{Z}/2\mathbb{Z} \subset \prod_{i\in J_c^1, j\in J_i^{ss}} \mathbb{Z}/2\mathbb{Z} \times \prod_{i\in J_c^1, j\in J_i^{qs}} \mathbb{Z}/2\mathbb{Z} \times \prod_{i\in J_{nc}^1, j\in J_i^{ss}} \mathbb{Z}/2\mathbb{Z} \times \prod_{i\in J_{nc}^1, j\in J_i^{qs}} \mathbb{Z}/2\mathbb{Z}.$$

Here, the last $\mathbb{Z}/2\mathbb{Z}$ embeds diagonally into $\prod_{i\in J_{nc}^1, j\in J_i^{qs}} \mathbb{Z}/2\mathbb{Z}$.

Proof. Let

$$G^{1,\mathrm{der}} = \prod_{i\in J^1} R_{F_i/\mathbb{Q}} H_i, \quad G^{2,\mathrm{der}} = \prod_{i\in J^2} R_{F_i/\mathbb{Q}} H_i, \quad G_c^{1,\mathrm{der}} = \prod_{i\in J_c^1} R_{F_i/\mathbb{Q}} H_i, \quad G_{nc}^{1,\mathrm{der}} = \prod_{i\in J_{nc}^1} R_{F_i/\mathbb{Q}} H_i,$$

and G^1, G^2, G_c^1, G_{nc}^1 be the associated reductive groups over $\mathbb{Q}$. By Lemma 3.2, we have

$$\pi(G^{\mathrm{der}}, G) = \pi(G^{1,\mathrm{der}}, G^1). \tag{3.1}$$

Case 1: Let $i \in J_c^1$. Since K_i/F_i splits over p, $R_{F_i/\mathbb{Q}}(R^{(1)}_{K_i/F_i}\mathbb{G}_{\mathrm{m}}) \otimes \breve{\mathbb{Q}}_p = \prod_{j\in J_i} \mathbb{G}_{\mathrm{m}}$. Base-changing over $\breve{\mathbb{Q}}_p$, we have

$$\prod_{i\in J_c} R_{F_i/\mathbb{Q}} H_i \to \prod_{i\in J_c} R_{F_i/\mathbb{Q}}(H_i \cdot R^{(1)}_{K_i/F_i}\mathbb{G}_{\mathrm{m}}) \rightsquigarrow \prod_{i\in J_c^1, j\in J_i} R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij} \to \prod_{i\in J_c^1, j\in J_i} R_{F_{ij}/\breve{\mathbb{Q}}_p}(H_{ij} \cdot \mathbb{G}_{\mathrm{m}}),$$

where $H_{ij} \cap \mathbb{G}_{\mathrm{m}} = \mu_2$. In particular, $H_{ij} \to H_{ij} \cdot \mathbb{G}_{\mathrm{m}}$ is $\mathrm{SO}_{2n_i+2} \to \mathrm{GSO}_{2n_i+2}$.

By construction, $G_c^{1,\mathrm{der}} \to G$ factors through $G_c^{1,\mathrm{der}} \cdot Z^c$, where $G_c^{1,\mathrm{der}} \cap Z^c = \prod_{i\in J_c} R_{F_i/\mathbb{Q}}\mu_2$. By Examples 2.3 and 2.6 and by Shapiro's lemma, $\ker(\pi_1(R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij})_I \to \pi_1(R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij} \cdot \mathbb{G}_\mathrm{m})_I)$ is trivial when $j \in J_i^{ss}$ and is the full $\mathbb{Z}/2\mathbb{Z}$ when $j \in J_i^{qs}$.

Case 2: Let $i \in J_{nc}^1$; we do not have a Z^c-factor. For $(?) = ss, qs$, denote

$$G_{nc,(?)}^{1,\mathrm{der}} := \prod_{i\in J_{nc}^1, j\in J_i^{(?)}} R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij}, \quad G_{nc,(?)}^1 := \left(\prod_{i\in J_{nc}^1, j\in J_i^{(?)}} R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij}\right) \cdot \mathbb{G}_\mathrm{m},$$

$$\widetilde{G}_{nc,(?)}^1 = \prod_{i\in J_{nc}^1, j\in J_i^{(?)}} \left((R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij}) \cdot \mathbb{G}_\mathrm{m}\right), \quad \widehat{G}_{nc,(?)}^1 = \prod_{i\in J_{nc}^1, j\in J_i^{(?)}} R_{F_{ij}/\breve{\mathbb{Q}}_p} (H_{ij} \cdot \mathbb{G}_\mathrm{m}),$$

where each $\mathbb{G}_\mathrm{m}$ intersects the group before it in the diagonal μ_2. We have

$$\mathbb{G}_\mathrm{m} \to \prod \mathbb{G}_\mathrm{m} \to \prod R_{F_{ij}/\breve{\mathbb{Q}}_p} \mathbb{G}_\mathrm{m} \rightsquigarrow G_{nc,(?)}^{1,\mathrm{der}} \to G_{nc,(?)}^1 \to \widetilde{G}_{nc,(?)}^1 \to \widehat{G}_{nc,(?)}^1.$$

Since $G_{nc,ss}^{1,\mathrm{der}} \to \widehat{G}_{nc,ss}^1$ is the $\prod$ of the Weil restriction of $H_{ij} \to H_{ij} \cdot \mathbb{G}_\mathrm{m}$, where H_{ij} is a split SO_{2n_i+2} and $\mathbb{G}_\mathrm{m} \cap H_{ij} = \mu_2$, $H_{ij} \cdot \mathbb{G}_\mathrm{m}$ is a split GSO_{2n_i+2}. By Example 2.6, $\pi_1(G_{nc,ss}^{1,\mathrm{der}})_I \to \pi_1(\widehat{G}_{nc,ss}^1)_I$ is injective; thus $\pi_1(G_{nc,ss}^{1,\mathrm{der}})_I \to \pi_1(G_{nc,ss}^1)_I$ is injective. Since $G_{nc,\breve{\mathbb{Q}}_p}^{1,\mathrm{der}} = G_{nc,ss}^{1,\mathrm{der}} \times G_{nc,qs}^{1,\mathrm{der}}$, by Lemma 3.3 (1),

$$\pi(G_{nc,\breve{\mathbb{Q}}_p}^{1,\mathrm{der}}, G_{nc,\breve{\mathbb{Q}}_p}^1) = \pi(G_{nc,qs}^{1,\mathrm{der}}, G_{nc,qs}^1). \tag{3.2}$$

Finally, for $i \in J_{nc}^1$, $j \in J_i^{qs}$, H_{ij} are quasi-split non-split SO_{2n_i+2} over F_{ij}. Let $\overline{H}_{ij}$ be a quasi-split non-split SO_{2n_i+2} over $\breve{\mathbb{Q}}_p$ with a splitting field E_{ij}, such that H_{ij} splits over $E_{ij} \cdot F_{ij}$ ($E_{ij} \cap F_{ij} = \breve{\mathbb{Q}}_p$), then we have a morphism $\overline{H}_{ij} \to R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij}$. By Shapiro's lemma, $\pi_1(\overline{H}_{ij})_I \xrightarrow{\sim} \pi_1(H_{ij})_{I_{ij}} = \pi_1(R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij})_I$. Denote

$$\overline{G}_{nc,qs}^{1,\mathrm{der}} := \prod_{i\in J_{nc}^1, j\in J_i^{qs}} \overline{H}_{ij}, \quad \overline{G}_{nc,qs}^1 := \left(\prod_{i\in J_{nc}^1, j\in J_i^{qs}} \overline{H}_{ij}\right) \cdot \mathbb{G}_\mathrm{m},$$

where $\mathbb{G}_\mathrm{m}$ intersects the product in diagonal μ_2. By Lemma 3.4, we have

$$\ker\left(\pi_1(G_{nc,qs}^{1,\mathrm{der}})_I \to \pi_1(G_{nc,qs}^1)_I\right) = \ker\left(\pi_1(\overline{G}_{nc,qs}^{1,\mathrm{der}})_I \to \pi_1(\overline{G}_{nc,qs}^1)_I\right). \tag{3.3}$$

Lemma 3.6 calculates the right hand side of Equation (3.3) explicitly. We are done by combining Lemma 3.8, Equation (3.1) and Equation (3.2). ☐

Lemma 3.8. *With the notation in Proposition 3.7, we have*

$$\ker\left(\pi_1(G_{\breve{\mathbb{Q}}_p}^{1,\mathrm{der}})_I \to \pi_1(G_{\breve{\mathbb{Q}}_p}^1)_I\right) = \ker\left(\pi_1(G_{c,\breve{\mathbb{Q}}_p}^{1,\mathrm{der}})_I \to \pi_1(G_{c,\breve{\mathbb{Q}}_p}^1)_I\right) \times \ker\left(\pi_1(G_{nc,\breve{\mathbb{Q}}_p}^{1,\mathrm{der}})_I \to \pi_1(G_{nc,\breve{\mathbb{Q}}_p}^1)_I\right).$$

Proof. Let $\widetilde{G}_c^1 = G_c^{1,\mathrm{der}} \cdot Z^c$. We first claim $\pi_1(\widetilde{G}_{c,\breve{\mathbb{Q}}_p}^1)_I \to \pi_1(G_{c,\breve{\mathbb{Q}}_p}^1)_I$ is injective. We have

$$\widetilde{G}_{c,\breve{\mathbb{Q}}_p}^1 \to G_{c,\breve{\mathbb{Q}}_p}^1 \rightsquigarrow \prod_{i\in J_c^1, j\in J_i} R_{F_{ij}/\breve{\mathbb{Q}}_p} (H_{ij} \cdot \mathbb{G}_\mathrm{m}) \to \left(\prod_{i\in J_c^1, j\in J_i} R_{F_{ij}/\breve{\mathbb{Q}}_p} H_{ij} \cdot \mathbb{G}_\mathrm{m}\right) \mathbb{G}_\mathrm{m},$$

where $\mathbb{G}_\mathrm{m}$ intersects the product in diagonal μ_2. Note that $H_{ij} \cdot \mathbb{G}_\mathrm{m} = \mathrm{GSO}_{2n_i+2}$. Consider $(H_{ij} \cdot \mathbb{G}_\mathrm{m}) \cdot \mathbb{G}_\mathrm{m}$, where $\mathbb{G}_\mathrm{m} \cap \mathrm{GSO}_{2n_i+2} = \mu_2$. Let $\widehat{G}_{c,\breve{\mathbb{Q}}_p}^1 = \prod_{i\in J_c^1, j\in J_i} R_{F_{ij}/\breve{\mathbb{Q}}_p} ((H_{ij} \cdot \mathbb{G}_\mathrm{m}) \cdot \mathbb{G}_\mathrm{m})$. Under the embedding $\mathbb{G}_\mathrm{m} \to \prod \mathbb{G}_\mathrm{m} \to \prod R_{F_{ij}/\breve{\mathbb{Q}}_p} \mathbb{G}_\mathrm{m}$, $\widetilde{G}_{c,\breve{\mathbb{Q}}_p}^1 \to \widehat{G}_{c,\breve{\mathbb{Q}}_p}^1$ is the $\prod$ of the Weil restrictions of $H_{ij} \cdot \mathbb{G}_{\mathrm{m}} \to (H_{ij} \cdot \mathbb{G}_{\mathrm{m}}) \cdot \mathbb{G}_{\mathrm{m}}$. By Lemma 3.3 (2) and 3.5, $\pi_1(\widetilde{G}^1_{c,\breve{\mathbb{Q}}_p})_I \to \pi_1(\widehat{G}^1_{c,\breve{\mathbb{Q}}_p})_I$ is injective. Since $\widetilde{G}^1_{c,\breve{\mathbb{Q}}_p} \to \widehat{G}^1_{c,\breve{\mathbb{Q}}_p}$ factors through $G^1_{c,\breve{\mathbb{Q}}_p}$, we proved the claim.

Now, let $\widetilde{G}^1 := G^{1,\mathrm{der}} \cdot Z^c$. Then $G^1 = \widetilde{G}^1 \cdot \mathbb{G}_{\mathrm{m}}$, $\widetilde{G}^1 = \widetilde{G}^1_c \times G^1_{nc}$. By Lemma 3.3 (1),

$$\ker\left(\pi_1(\widetilde{G}^1_{\breve{\mathbb{Q}}_p})_I \to \pi_1(G^1_{\breve{\mathbb{Q}}_p})_I\right) = \ker\left(\pi_1(G^{1,\mathrm{der}}_{nc,\breve{\mathbb{Q}}_p})_I \to \pi_1(G^1_{nc,\breve{\mathbb{Q}}_p})_I\right).$$

Since $G^{1,\mathrm{der}} \to G^1$ factors through $\widetilde{G}^1$, and $G^{1,\mathrm{der}} \to \widetilde{G}^1$ is the product of $G^{1,\mathrm{der}}_c \to \widetilde{G}^1_c$ and $G^{1,\mathrm{der}}_{nc} \to G^1_{nc}$, we are done. □

4. MODIFIED CONSTRUCTION

This part is [MW26, Prop. 6.18- Lem. 6.21]

Construction 4.1. For an abelian-type (G_2, X_2), we associate a (G, X) as Proposition 2.1. Let $(T, h) \subset (G, X)$ be a special point with T a maximal torus. We first consider $G^{db} := G \times_{G^{ab}} G$ where the map $G \to G^{ab} = G/G^{\mathrm{der}}$ is the natural one. Denote $T^{ab} := T/(T \cap G^{\mathrm{der}})$. Define $G^{rf} := G \times_{G^{ab}} T$.

Then $G^{db} \cong (G^{\mathrm{der}} \times G^{\mathrm{der}}) \cdot Z^c \cdot \mathbb{G}_m \subset G \times G$.

In the second expression of the above line, $Z^c \cdot \mathbb{G}_m$ maps to $G \times G$ diagonally into $(Z^c \cdot \mathbb{G}_m) \times (Z^c \cdot \mathbb{G}_m)$.

Denote the corresponding Shimura data by (G^{db}, X^{db}) and (G^{rf}, X^{rf}), respectively. Then both of them are of Hodge type because they are both contained in the Hodge-type Shimura datum defined by $G \times_{\eta, \mathbb{G}_m, \eta} G$. The shortcoming of this construction is that $Z(G^{rf})$ is not connected.

Let us show that

Proposition 4.2. *Given an abelian-type (G_2, X_2), the accessible Hodge-type lifting in the sense of* [MW26, Def. 6.16] *is (G^{rf}, X^{rf}), as given by Construction 4.1.*

More precisely, $\pi(G^{rf,\mathrm{der}}, G^{rf})$ is trivial. In particular, the intersection of any parahoric subgroup $\breve{K}_p$ of $G^{rf}(\breve{\mathbb{Q}}_p)$ with $G^{rf,\mathrm{der}}(\breve{\mathbb{Q}}_p)$ is parahoric.

Proof. It suffices to show that $\pi(G^{rf,\mathrm{der}}, G^{rf})$ is trivial. We first consider $\pi(G^{db,\mathrm{der}}, G^{db})$. By the computation of Proposition 3.7 and Lemma 4.3, we have that

$$\pi(G^{db,\mathrm{der}}, G^{db}) = \mathrm{diag}_{\{1,2\}}\{\prod_{i \in J^1_c, j \in J^{qs}_i} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\},$$

where the index $\{1,2\}$ labels the two factors of $G^{db,\mathrm{der}} = G^{\mathrm{der}} \times G^{\mathrm{der}}$.

Note that $G^{rf,\mathrm{der}} = G^{\mathrm{der}} \times \{1\} \subset G^{db,\mathrm{der}}$. Then $\pi(G^{\mathrm{der}} \times \{1\}, G^{db}) \subset \pi(G^{db,\mathrm{der}}, G^{db})$ is trivial. So $\pi(G^{rf,\mathrm{der}}, G^{rf}) \subset \pi(G^{rf,\mathrm{der}}, G^{db})$ is trivial. □

Lemma 4.3. *Let G be any reductive group. Write $\pi : G \to G^{ab}$ and $G^{db} := G \times_{G^{ab}} G$. Then*

$$\pi(G^{db,\mathrm{der}}, G^{db}) = \mathrm{diag}_{\{1,2\}}\, \pi(G^{\mathrm{der}}, G) \subset \pi(G^{\mathrm{der}} \times G^{\mathrm{der}}, G \times G).$$

Proof. Consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
 & G^{\mathrm{der}} \times G^{\mathrm{der}} \arrow[r, equal] \arrow[d, hook] & G^{\mathrm{der}} \times G^{\mathrm{der}} \arrow[d, hook] & & \\
0 \arrow[r] & G^{db} \arrow[r, "i"] \arrow[d, two heads, "\pi"] & G \times G \arrow[r, no head, "{(\pi,\pi^{-1})}"] \arrow[d, two heads] & G^{ab} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G^{ab} \arrow[r, "\Delta"] & G^{ab} \times G^{ab} \arrow[r, no head, "{(\mathrm{id},\mathrm{id}^{-1})}"] & G^{ab} \arrow[r] & 0.
\end{tikzcd}$$

Taking the long exact sequences of it, we find

$$\begin{tikzcd}
H_1(I,\pi_1(G^{ab})_I) \arrow[r, hook, "\Delta"] \arrow[d, "\Delta(\delta)"] & H_1(I,\pi_1(G^{ab})_I)\times H_1(I,\pi_1(G^{ab})_I) \arrow[d, "{(\delta,\delta)}"] \\
\pi_1(G^{\mathrm{der}})_I\times\pi_1(G^{\mathrm{der}})_I \arrow[r, dash, "="] \arrow[d] & \pi_1(G^{\mathrm{der}})_I\times\pi_1(G^{\mathrm{der}})_I \arrow[d] \\
\pi_1(G^{db})_I \arrow[r, "i"] & \pi_1(G)_I\times\pi_1(G)_I.
\end{tikzcd}$$

From the diagram, we see that $\pi(G^{db,\mathrm{der}}, G^{db}) = \operatorname{im}\Delta(\delta)$ and $\pi(G^{\mathrm{der}}\times G^{\mathrm{der}}, G\times G) = \operatorname{im}\delta\times\delta$. It follows from the definition that $\operatorname{im}\Delta(\delta)\subset\operatorname{im}\delta\times\delta$ is exactly $\operatorname{diag}_{\{1,2\}}\operatorname{im}\delta = \operatorname{diag}_{\{1,2\}}\pi(G^{\mathrm{der}}, G)$. $\square$

Now let (G, X) be any Hodge lifting of (G_2, X_2), then $G^{db}\subset G\times_{\eta,\mathbb{G}_{\mathrm{m}},\eta}G$, where $\eta : G\to\mathbb{G}_{\mathrm{m}}$ is the similitude character (here η factors through G^{ab}), then (G^{db}, X^{db}) is a Hodge-type Shimura datum. Therefore, (G^{rf}, X^{rf}) is a Hodge-type lifting of (G_2, X_2) and satisfies the wanted proposition.

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MORNINGSIDE CENTER OF MATHEMATICS, BEIJING 100190, CHINA
Email address: `maoshengkai@amss.ac.cn`

BICMR, PEKING UNIVERSITY, BEIJING 100871, CHINA
Email address: `wuph@pku.edu.cn`